Composing Codensity Bisimulations

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Coalgebraic bisimulation [Hermida & Jacobs, Inf. Comput. '98]

$$\begin{tikzcd} \dot{F} \circlearrowright \underbrace{\mathbb{E}}_{\text{Sets w/ spatial structure}} \arrow[d, "\text{fibration } p"'] \\ F \circlearrowright \underbrace{\mathbb{B}}_{\text{Sets}} \end{tikzcd} \qquad \begin{tikzcd} \underbrace{P \longrightarrow \dot{F}P}_{\text{bisimulation}} \\ \underbrace{X \longrightarrow FX}_{\text{transition system}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Coalg}(\dot{F}) \arrow[d, "\mathrm{Coalg}(p)"] \\ \mathrm{Coalg}(F) \end{tikzcd}$$

cat. of F-bisimulations

Cat. of F-coalgs.

e.g.

- Standard bisimulation on a Kripke frame δ

$$\begin{tikzcd} \dot{P} \circlearrowright \mathbf{ERel} \arrow[d] \\ P \circlearrowright \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \underbrace{R \longrightarrow \dot{P}R} \\ S \xrightarrow{\delta} PS \end{tikzcd}$$

$\Leftrightarrow$ equiv.

$R \subseteq S \times S$ is a bisimulation on δ

$$\left(\forall (x,y) \in R. \begin{cases} x' \in \delta x \Rightarrow \exists y' \in \delta y.\ (x', y') \in R \\ y' \in \delta y \Rightarrow \exists x' \in \delta x.\ (x', y') \in R \end{cases} \right)$$

- language equivalence of det. automata
- bisimilarity metric for MDP.

Composing bisimulations via lifting of distributive law

[Bonchi+, Act. info.'17]

Compose bisimulations

by liftings $\dot{F}, \dot{T}, \dot{\lambda}$ of F, T, λ along p.

by $F : \mathbb{B} \to \mathbb{B}$, $T : \mathbb{B}^N \to \mathbb{B}$, distributive law $\lambda : TF^N \Rightarrow FT$

$$
\begin{tikzcd}
\mathrm{Coalg}(\dot{F})^N \arrow[r, "\dot{T}_{\dot{\lambda}}"] \arrow[d] & \mathrm{Coalg}(\dot{F}) \arrow[d, "\mathrm{Coalg}(p)"] \\
\mathrm{Coalg}(F)^N \arrow[r, "T_\lambda"] & \mathrm{Coalg}(F)
\end{tikzcd}
$$

Compose systems

where
$\dot{T}_{\dot{\lambda}} = \dot{\lambda} \circ \dot{T}(-)$
$T_\lambda = \lambda \circ T(-)$

<u>e.g.</u> product composition of standard bisimulations.

T_λ … product of Kripke frames. δ_1, δ_2

$\dot{T}_{\dot{\lambda}}$ … product of bisimulations R_1, R_2

<u>Prop.</u>
$$
\left\{
\begin{array}{l}
\underbrace{R_1}_{\text{bisim on } \delta_i} \in \mathrm{Coalg}(\dot{F})_{\delta_1} \\
R_2 \in \mathrm{Coalg}(\dot{F})_{\delta_2}
\end{array}
\right.
\Rightarrow
\underbrace{\overbrace{\dot{T}_{\dot{\lambda}}(R_1, R_2)}^{= R_1 \times R_2}}_{\text{bisim on } T_\lambda(\delta_1, \delta_2)} \in \mathrm{Coalg}(\dot{F})_{T_\lambda(\delta_1, \delta_2)}
$$

Overview of our work

$$\begin{tikzcd}
\mathrm{Coalg}(\dot F)^{N} \arrow[r, "\dot T_\lambda"] \arrow[d] & \mathrm{Coalg}(\dot F) \arrow[d] \\
\mathrm{Coalg}(F)^{N} \arrow[r, "T_\lambda"] & \mathrm{Coalg}(F)
\end{tikzcd}$$

] given by liftings $\underbrace{\dot F, \dot T}_{\text{defined by Codensity liftings.}}, \underbrace{\dot\lambda}_{\text{ensured by a suf. cond.}}$

] given by F, T, λ

We adopt { Codensity lifting for $\dot F$ [Sprungert, JLC'21]
generalized codensity lifting for $\dot T$ new!

and provide a sufficient condition of liftability of λ
that gives various composition of bisimulations.

Results are based on 2-categorical extension of Beohar's decomposition.
[Beohar, Gurke, König, Messing, Forster, Schröder, Wild, STACS'24]

Comparison to related works.

1. Abstract GSOS rules. [Turi & Plotkin. LICS'97] et al.
   not directly apply to similarity or behavioural metric
   ↑ Our results cover.

2. Liftability of λ in [Bonchi+. Act. info.'17]

	[§6, Bonchi+'17]	[§8, Bonchi+'17]	ours.
fibration p	$\mathbf{Rel} \to \mathbf{Set}$	✓ general various	Sweet spot! various
$\dot{F}, \dot{T}$	$\mathrm{Rel}(F)$, Rfl, Sym, ...	various	Codensity liftings
proof of liftability	✓ good cond. all distributive laws lift	case-by-case	by sufficient cond.

Outline

1. Codensity lifting.

   - Ordinary one.

$$\begin{tikzcd} {[F,\tau]^\Omega} \circlearrowright \mathbb{E} \arrow[d, "p"] \\ F \circlearrowright \mathbb{B} \end{tikzcd}$$

   … for system's behavior

   - Generalizing Codensity lifting.

$$\begin{tikzcd} \mathbb{E}^N \arrow[r, "{[T,\sigma]^\Omega}"] \arrow[d, "p^N"'] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B}^N \arrow[r, "T"'] & \mathbb{B} \end{tikzcd}$$

   … for composing coalg carriers

2. Sufficient condition of liftability of distributive laws.

$$\begin{tikzcd} {[T,\sigma]^\Omega\left(([F,\tau]^\Omega)^N\right)} \arrow[r, Rightarrow, "?"] \arrow[d, phantom, "\vdots"] & {[F,\tau]^\Omega [T,\sigma]^\Omega} \\ TF^N \arrow[r, Rightarrow, "\lambda"] & FT \end{tikzcd}$$

Codensity lifting $[F,\tau]^\Omega$

[Sprunger+, JLC'21]

<u>Def</u>.

Given
$$\left[\begin{array}{cc} \begin{array}{c}\mathbb{E}\\ \text{clatn-fib. } p\big\downarrow \\ F \circlearrowright \mathbb{C}\end{array} & \begin{array}{c}\text{truth value}\\ \Omega \in \mathbb{E}_\Omega \\ \\ F\Omega \xrightarrow{\tau} \Omega \\ \text{modality}\end{array}\end{array}\right],$$

$[F,\tau]^\Omega : \mathbb{E} \to \mathbb{E}$ is the largest lifting of F s.t. τ is liftable.

$$\begin{array}{cc} [F,\tau]^\Omega \circlearrowright \mathbb{E} & [F,\tau]^\Omega \Omega \xrightarrow{\dot\tau} \Omega \\ \big\downarrow p & \\ F \circlearrowright \mathbb{C} & F\Omega \xrightarrow{\tau} \Omega. \end{array}$$

Kantorovich style

It can be written as $[F,\tau]^\Omega(P) = \bigwedge_{k \in_p \mathbb{E}(P,\Omega)} (\tau \circ Fk)^* \Omega$

e.g. Lifting for standard bisim, Kantorovich metric, ...

Generalizing codensity lifting $[F, \tau]^{\Omega, \Pi}$

original Beohar's decomposition [Beohart. STACS'24]

$$\begin{array}{l} \mathbb{E} \ni [F,\tau]^{\Omega} = \underset{\bigwedge (-)^{*}\Omega}{\underset{\shortparallel}{R^{p,\Omega}}} \circ \underset{\tau \circ F(-)}{\underset{\shortparallel}{Sp(F,\tau)}} \circ \underset{R \in p\mathbb{E}(-,\Omega)}{\underset{\shortparallel}{L^{p,\Omega}}} \quad \left(= \bigwedge_{R \in p\mathbb{E}(-,\Omega)} (\tau \circ FR)^{*}\Omega \right) \\ \quad p \big\downarrow \\ \mathbb{B} \ni F \end{array}$$

<u>Def.</u>

$$\begin{tikzcd} \mathbb{E} \arrow[r, dashed, "{[F,\tau]^{\Omega,\Pi}}"] \arrow[d, "p"] & \mathbb{F} \arrow[d, "q"] \\ \mathbb{B} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

truth values

Given $\Omega \in \mathbb{E}_{\Omega}$, $\Pi \in \mathbb{F}_{\Pi}$,

modality

$\tau : F\Omega \to \Pi$ in $\mathbb{C}$.

2-functor
$Sp : 1 /\!/ CAT \to CAT^{CLatn}$

$$[F,\tau]^{\Omega,\Pi} := \underset{\bigwedge (-)^{*}\Pi}{\underset{\shortparallel}{R^{q,\Pi}}} \circ \underset{\tau \circ F(-)}{\underset{\shortparallel}{Sp(F,\tau)}} \circ \underset{R \in p\mathbb{E}(-,\Omega)}{\underset{\shortparallel}{L^{p,\Omega}}} \quad \left(= \bigwedge_{R \in p\mathbb{E}(-,\Omega)} (\tau \circ FR)^{*}\Pi \right)$$

<u>Thm.</u> $[F,\tau]^{\Omega,\Pi}$ is the largest lifting of F s.t. τ is liftable.

Outline

1. Codensity lifting.
   - Ordinary one.

$$
\begin{tikzcd}
\mathbb{E} \arrow[loop left, "{[F,\tau]^\Omega}"] \arrow[d, "p"] \\
\mathbb{B} \arrow[loop left, "F"]
\end{tikzcd}
$$

… for system's behavior

   - Generalizing Codensity lifting.

$$
\begin{tikzcd}
\mathbb{E}^N \arrow[r, "{[T,\sigma]^\Omega}"] \arrow[d, "p^N"'] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{B}^N \arrow[r, "T"'] & \mathbb{B}
\end{tikzcd}
$$

… for composing coalg carriers

2. Sufficient condition of liftability of distributive laws.

$$[T,\sigma]^\Omega\left([F,\tau]^\Omega\right)^N \overset{?}{\Rightarrow} [F,\tau]^\Omega [T,\sigma]^\Omega$$

$$\vdots$$

$$TF^N \overset{\lambda}{\Rightarrow} FT$$

Liftability of distributive law λ

generalized one

ordinary one

$$\begin{tikzcd} \mathbb{E}^N \arrow[r, "{[T,\sigma]^{\Omega^N,\Omega}}"] \arrow[d, "p^N"'] & \mathbb{E} \arrow[loop right, "{[F,\tau]^{\Omega}}"] \arrow[d, "p"] \\ \mathbb{B}^N \arrow[r, "T"] & \mathbb{B} \arrow[loop right, "F"] \end{tikzcd}$$

Consider $\Omega \in \mathbb{E}_\Omega$,

$\tau : F\Omega \to \Omega$ in $\mathbb{B}$,

$\sigma : T(\Omega, \cdots, \Omega) \to \Omega$ in $\mathbb{B}$,

We tackle the problem:

when $\lambda : TF^N \Rightarrow FT$ is liftable to

$\dot{\lambda} : [T,\sigma]^{\Omega}([F,\tau]^{\Omega})^N \Rightarrow [F,\tau]^{\Omega}[T,\sigma]^{\Omega}$?

It gives

compose bisimulations

$$\begin{tikzcd} \mathrm{Coalg}([F,\tau]^{\Omega})^N \arrow[r, "{[T,\sigma]^{\Omega}_{\dot{\lambda}}}"] \arrow[d] & \mathrm{Coalg}([F,\tau]^{\Omega}) \arrow[d, "\mathrm{Coalg}(p)"] \\ \mathrm{Coalg}(F)^N \arrow[r, "T_\lambda"] & \mathrm{Coalg}(F) \end{tikzcd}$$

compose systems

Our Sufficient Condition

How do we lift λ to $[T,\tau]^{\Omega}([F,\tau]^{\Omega})^{N} \Rightarrow [F,\tau]^{\Omega}[T,\tau]^{\Omega}$?

$$
\begin{aligned}
& [T,\tau]^{\Omega}([F,\tau]^{\Omega})^{N} \\
&= [T,\tau]^{\Omega}[F^{N},\tau^{N}]^{\Omega} \\
&= R \circ Sp(T,\tau) \circ L \circ R \circ Sp(F^{N},\tau^{N}) \circ L \\
&\Rightarrow R \circ \underline{Sp(T,\tau) \circ Sp(F^{N},\tau^{N})} \circ L \qquad \text{by } L \dashv R \\
&\overset{(1)}{\Rightarrow} R \circ \underline{Sp(F,\tau) \circ Sp(T,\tau)} \circ L \\
&\overset{(2)}{=} [F,\tau]^{\Omega}[T,\tau]^{\Omega}
\end{aligned}
$$

<u>Thm.</u> $\lambda : TF^{N} \Rightarrow FT$ is liftable if (1) and (2) hold.

(1) λ is a 2-cell $(T,\tau)\circ(F^{N},\tau^{N}) \Rightarrow (F,\tau)\circ(T,\tau)$ in $1/\!/\mathbf{CAT}$.

(2) The last equality holds.

(iff $Sp(T,\tau)\circ LP$ is approximating to $[F,\tau]^{\Omega}$ for each P)
in the sense of [Komorida+, LICS'21]

Examples of composition of bisimulations

<table>
<tr><th></th><th>Composition of systems. ($T\lambda$)</th><th>Composition of bisim. ($[T,\sigma]^{\Omega}_{\dot\lambda}$)</th><th></th></tr>
<tr><td>Standard bisim.</td><td rowspan="5">parallel composition</td><td rowspan="2">product</td><td rowspan="4">satisfy our sufficient cond.</td></tr>
<tr><td>Lang. equivalence of det. automata</td></tr>
<tr><td>Bisimilarity pseudometric for det. automata.</td><td rowspan="2">max</td></tr>
<tr><td>Similarity "</td></tr>
<tr><td>Bisimulation metric for MDP</td><td>given by $\sigma(a,b) = 1-(1-a)(1-b)$</td><td>unsolved if it satisfies.</td></tr>
</table>

Conclusions

$$\begin{tikzcd}
\mathrm{Coalg}([F,\tau]^{\Omega})^{N} \arrow[r, "{[T,\sigma]^{\Omega}_{\dot\lambda}}"] \arrow[d] & \mathrm{Coalg}([F,\tau]^{\Omega}) \arrow[d, "\mathrm{Coalg}(p)"] \\
\mathrm{Coalg}(F)^{N} \arrow[r, "T_{\lambda}"] & \mathrm{Coalg}(F)
\end{tikzcd}$$

compose bisimulations

compose systems

- Generalizing codensity liftings for $[T,\sigma]^{\Omega} : \mathbb{E}^{N} \to \mathbb{E}$

- A sufficent condition of liftability
  - commutation b/w τ and σ.
  - approximation property.

$$[T,\sigma]^{\Omega}([F,\tau]^{\Omega})^{N} \overset{\dot\lambda}{\Rightarrow} [F,\tau]^{\Omega}[T,\sigma]^{\Omega}$$
$$\vdots$$
$$TF^{N} \overset{\lambda}{\Rightarrow} FT$$

- (omitted in this talk) A composition of codensity games via modalities. It preserves invariants under our sufficient condition.

Appendix

Example: Composing bisimulations for Kripke frames

Def. A bisimulation on a Kripke frame $\delta : S \to \mathcal{P}S$ is $R \subseteq S \times S$ s.t.

$$\left(\begin{tikzcd} x \arrow[r] \arrow[d, dotted, "R"] & x' \\ y & \end{tikzcd} \Rightarrow \begin{tikzcd} x \arrow[r] \arrow[d, dotted, "R"] & x' \arrow[d, dotted, "R"] \\ y \arrow[r] & \exists y' \end{tikzcd}\right) \text{ and } \left(\begin{tikzcd} x \arrow[d, dotted, "R"] & \\ y \arrow[r] & y' \end{tikzcd} \Rightarrow \begin{tikzcd} x \arrow[r] \arrow[d, dotted, "R"] & \exists x' \arrow[d, dotted, "R"] \\ y \arrow[r] & y' \end{tikzcd}\right)$$

Prop. R_1 and R_2 are bisim. on δ_1 and δ_2, respectively.
$\Rightarrow R_1 \times R_2$ is a bisim. on the product of δ_1 and δ_2.

Coalgebraic reformulation

- Kripke frames δ_1, δ_2 $\longleftrightarrow$ Coalgebras c_1, c_2
  Compose by a product — Compose by a distributive law λ
- Bisimulations R_1, R_2 $\longleftrightarrow$ liftings of c_1, c_2 along a fibration.
  Compose by a product ($R_1 \times R_2$) — Compose by a lifting of λ. (our focus)

Composing coalgebras via distributive law

Given functors $F : \mathbb{B} \to \mathbb{B}$, $T : \mathbb{B}^N \to \mathbb{B}$,

a distributive law $\lambda : TF^N \Rightarrow FT$,

we have $T_\lambda :$ $\mathrm{Coalg}(F)^N \longrightarrow \mathrm{Coalg}(F)$

$$\underbrace{\{X_i \xrightarrow{c_i} FX_i\}_{i \in \{1,\dots,N\}}}_{\text{Component system}} \overset{T_\lambda}{\longmapsto} \underbrace{T\vec{X} \xrightarrow{T\vec{c}} T\vec{FX} \xrightarrow{\lambda_{\vec{X}}} F(T\vec{X})}_{\text{Composite system.}}$$

e.g. product of Kripke frames $\{c_i : X_i \to \mathcal{P}X_i\}_{i=1,2}$

$T := \times : \mathbf{Set}^2 \to \mathbf{Set}$. $\quad \lambda_{X,Y}(A,B) := \{(a,b) \mid a \in A, b \in B\}$

Then $T_\lambda(c_1, c_2) : X_1 \times X_2 \longrightarrow \mathcal{P}(X_1 \times X_2)$

$$a, b \longmapsto \{(a', b') \mid a' \in c_1 a,\ b' \in c_2 b\}$$

How do we construct a bisimulation on $\underbrace{T_\lambda(\vec{c})}_{\text{Composite system}}$?

Beohar et al.'s decomposition, Fibrationally [Beohar+, STACS'24]

decomposition of $[F,\tau]^{\Omega}$ as being sandwiched b/w adjoints.

Thm. $[F,\tau]^{\Omega} = \underset{=\,\bigwedge(-)^{*}\Omega}{R^{p,\Omega}} \circ \underset{=\,\tau\circ F(-)}{Sp(F,\tau)} \circ \underset{=\,k\in p\mathbb{E}(-,\Omega)}{L^{p,\Omega}} \quad \left(= \bigwedge_{k\in p\mathbb{E}(-,\Omega)} (\tau\circ Fk)^{*}\Omega \right)$

where

$$
\begin{tikzcd}
\mathbb{E} \arrow[dr, dashed, "L^{p,\Omega}"] \arrow[drr, bend left, "\mathbb{E}(-,\Omega)^{op}"] \arrow[ddr, bend right, "p"'] & & \\
& Sp(\mathbb{B},\Omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}\mathrm{red}^{op} \arrow[d] \\
& \mathbb{B} \arrow[r, "\mathbb{B}(-,\Omega)^{op}"'] & \mathbf{Set}^{op}
\end{tikzcd}
$$

fibred lifting of F

$$
\begin{tikzcd}
Sp(\mathbb{B},\Omega) \arrow[loop right, "Sp(F,\tau)"] \arrow[d, "\mathbf{CLat}_{\wedge}\text{-fib.}"] \\
\mathbb{B} \arrow[loop right, "F"]
\end{tikzcd}
$$

$Sp(F,\tau)(S\subseteq \mathbb{B}(X,\Omega)) := \{\tau\circ Fk \mid k\in S\} \subseteq \mathbb{B}(FX,\Omega)$

essential part given by F, τ

$R^{p,\Omega}$ is the right adjoint of $L^{p,\Omega}$.

Codensity liftings of T. (esp. product functors.)

<u>Example</u> Pseudo-metric lifting by a modality σ.

$$\begin{tikzcd} \mathbf{PMet}^2 \arrow[r, "{[X,\sigma]^{d_I}}"] \arrow[d] & \mathbf{PMet} \arrow[d] \\ \mathbf{Set}^2 \arrow[r, "X"] & \mathbf{Set} \end{tikzcd}$$

Euclidean distance.

$\Omega := d_I \in \mathbf{PMet}_{[0,1]}$

$\sigma : [0,1] \times [0,1] \longrightarrow [0,1]$ is a function.

Prop. $[X,\sigma]^{d_I}(d_X, d_Y)\,((x,y),(x',y')) = \underline{\sigma(d_X(x,x'),\, d_Y(y,y'))}$

if $\begin{cases} \sigma \text{ is monotone,} \\ \sigma(0,0)=0, \\ \sigma(a,b)-\sigma(c,d) \le \sigma(|a-c|, |b-d|) \text{ for each } a,b,c,d \in [0,1] \end{cases}$

e.g. $\sigma(a,b) := \underline{\dfrac{a+b}{2},\ \max(a,b),\ 1-(1-a)(1-b)}$

They give various non-trivial liftings.