

A No-go Theorem for Coalgebraic Product Construction

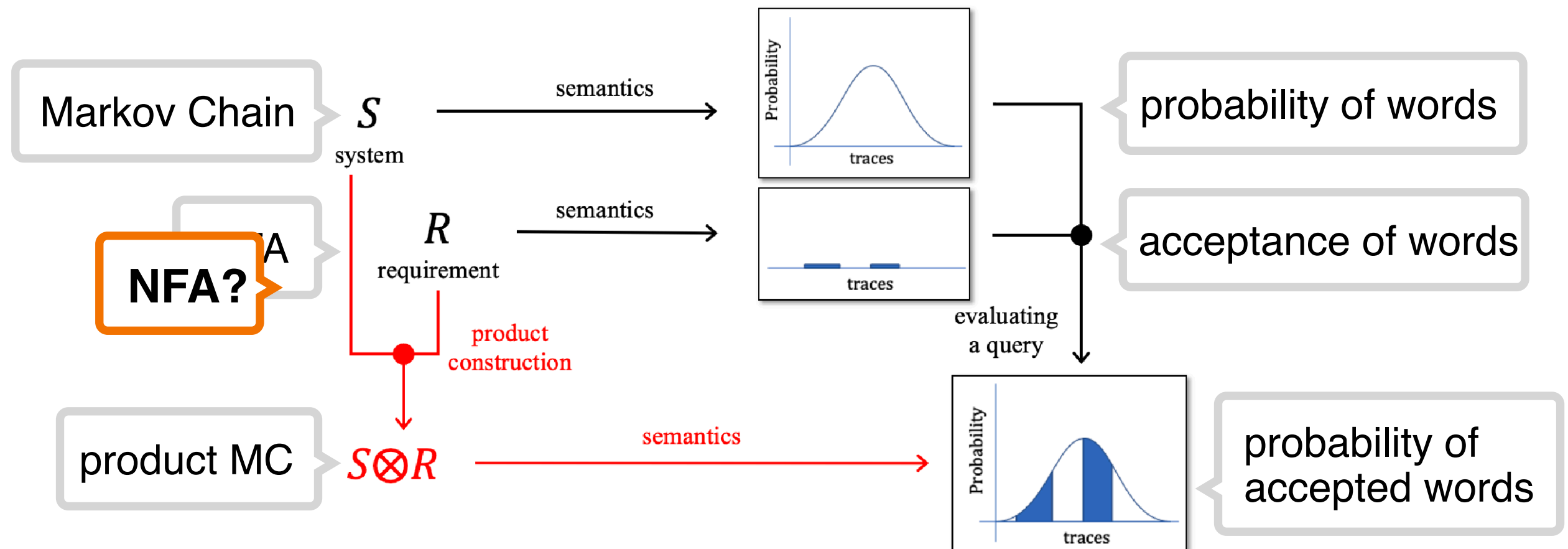
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FoSSaCS'26

Background

Product construction_{[Pnueli, FOCS'77],[VardiW, JCSS'86]}

Given a system and a requirement,
to what extent does traces satisfy the requirement?

e.g. Product MCs of MCs and DFAs



Background

Product construction for a class of NFA

Can we apply product constructions for MCs and NFAs?

→ No. (empirically)

- product constructions for MCs with
 - separate unambiguous automata [CouvreurSS,LPAR'03]
 - unambiguous automata [Baier+, '23]
 - limit-deterministic automata [CourcoubetisY'95, Sickert+CAV'16, Vardi FOCS'85]
- } between DFA and NFA

There is no formal proof of this impossibility as far as we know

Contributions

1. No-go theorem for product construction for MCs and NFAs for computing $\text{Pr}(\text{accepting traces})$.
2. New product construction for MCs and MFAs (multiset finite automata) for computing $\text{ExpNum}(\text{accepting traces})$.
3. Characterisation of natural transformations $F_A \Rightarrow F_B$ btw functors obtained from commutative monoids A, B

$F_A \backslash F_B$	$\mathcal{P}_f$	$\mathcal{M}$	$\mathcal{R}_{\geq 0}^+$	$\mathcal{R}_{\geq 0}^\times$
$\mathcal{P}_f$	2	1	1	2
$\mathcal{M}$	$\mathbf{2}^{\mathbb{N}_{\geq 1}}$	$\mathbb{N}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$
$\mathcal{R}_{\geq 0}^+$	$\mathbf{2}^{\mathbb{R}_{>0}}$	1	$\mathbb{R}_{\geq 0}^{\mathbb{R}_{>0}}$	?

Contribution 1

No-go theorem for product construction

Thm. There is no (coalgebraic) product MCs of MCs and NFAs for computing $\text{Pr}(\text{accepting traces})$.

Coalgebraic product construction [Watanabe+, OOPSLA1'25]

System $X \xrightarrow{c} F_S X$,
Requirement $Y \xrightarrow{d} F_R Y$ $\mapsto$ Product $X \times Y \xrightarrow{c \otimes_\lambda d} F_{S \otimes R}(X \times Y)$
s.t. $\llbracket c \otimes_\lambda d \rrbracket$ represents the target problem

a distributive law

$$\lambda: \times \circ (F_S \times F_R) \Rightarrow F_{S \otimes R} \circ \times$$

(well-behaved for semantics)

We show that there is **no** such λ for (MC, NFA $\mapsto$ MC).

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Contribution 2

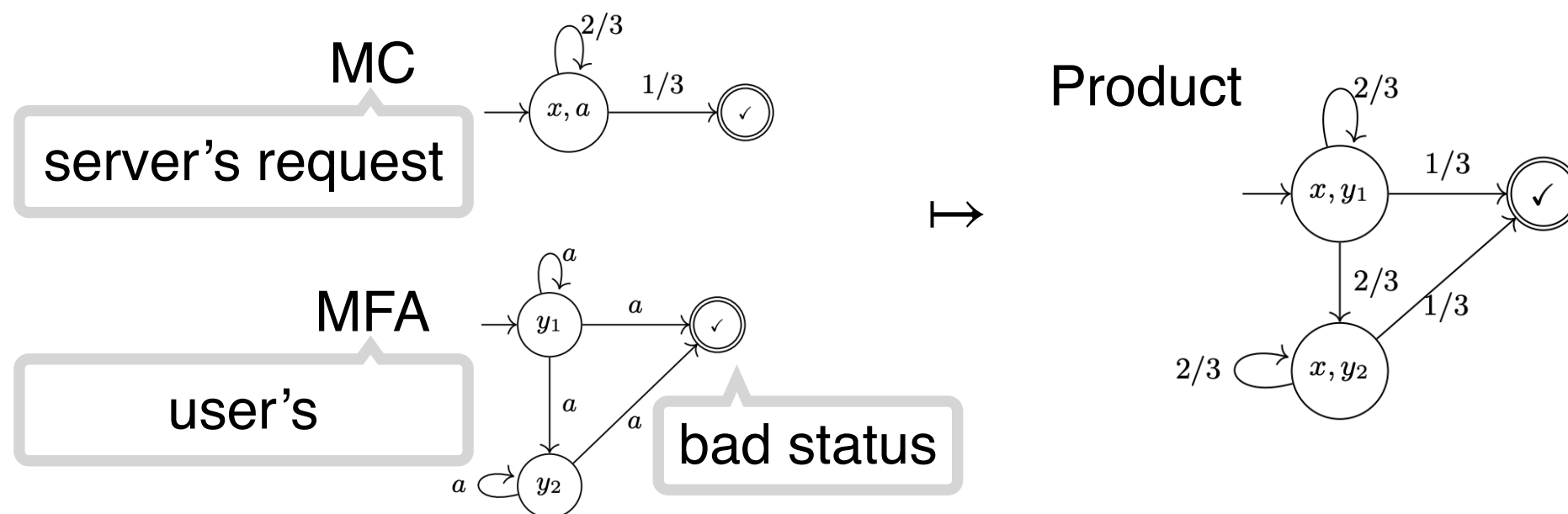
Product construction for MCs and MFAs

We present a new instance for multiset finite automata (MFAs):

Prop. There is a unique product construction for MCs and MFAs.

The target problem: the expected number of accepting traces
It is solvable in polynomial time.

e.g. What is the expected number of undesirable behaviours?



Contributions

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Contribution 3

Characterisation of natural trans.

Thm. No coalgebraic product MCs of MCs and NFAs
for computing $\text{Pr}(\text{accepting traces})$.
(i.e. there is no λ that is well-behaved for semantics).

Proof sketch. Using the result that $\mathcal{P} \Rightarrow \mathcal{D}_{\leq 1}$ uniquely exists, we restrict the form of $\{\lambda_{X,Y}: \mathcal{D}_{\leq 1}X \times A \times \mathcal{P}(Y \times 1)^A \rightarrow \mathcal{D}_{\leq 1}(X \times Y + 1)\}_{X,Y}$. We show that such λ does not satisfy the desired condition.

To discuss for DFAs/NFAs/MFAs, we characterise $F_A \Rightarrow F_B$.

$F_A \setminus F_B$	$\mathcal{P}_f$	$\mathcal{M}$	$\mathcal{R}_{\geq 0}^+$	$\mathcal{R}_{\geq 0}^\times$
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$\mathcal{M}$	$2^{\mathbb{N}_{\geq 1}}$	$\mathbb{N}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$
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It restricts the form of distributive laws λ .

**Characterisation of
natural trans. $F_A \Rightarrow F_B$**

Functors that arise from commutative monoids

Let **CMon** be the category of commutative monoids.

Def. $F_{(_)}: \mathbf{CMon} \rightarrow [\mathbf{Set}, \mathbf{Set}]$ is defined as follows.

For $(A, +, 0)$, $F_A(X) := \{h: X \rightarrow A \mid \text{supp}(h) \text{ is finite}\},$

$$F_A(f)(h) := \sum_{x \in f^{-1}(-)} h(x),$$

For $i: A \rightarrow B$, $F_i(f) := i \circ f.$

Example

- $F_{(\{\perp, \top\}, \vee, \perp)}$ is the finite powerset functor $\mathcal{P}_f$
- $F_{(\mathbb{N}, +, 0)}$ is the multiset functor $\mathcal{M}$, $R_{\geq 0}^+ := F_{(\mathbb{R}_{\geq 0}, +, 0)}$, $R_{\geq 0}^\times := F_{(\mathbb{R}_{\geq 0}, \times, 1)}$

Characterisation of $F_A \Rightarrow F_B$

We characterise $\text{Nat}(F_A, F_B)$

- $\mathcal{P}_f \Rightarrow \mathcal{P}_f$ and $\mathcal{M} \Rightarrow \mathcal{M}$
[Dahlqvist&Neves'18]

$F_A \setminus F_B$	$\mathcal{P}_f$	$\mathcal{M}$	$\mathcal{R}_{\geq 0}^+$	$\mathcal{R}_{\geq 0}^\times$
$\mathcal{P}_f$	2	1	1	2
$\mathcal{M}$	$\mathbf{2}^{\mathbb{N}_{\geq 1}}$	$\mathbb{N}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$
$\mathcal{R}_{\geq 0}^+$	$\mathbf{2}^{\mathbb{R}_{>0}}$	1	$\mathbb{R}_{\geq 0}^{\mathbb{R}_{>0}}$	?

(Non-gray entries are our new results)

For no-go theorem, we determine the form of natural trans.

Corollary 1. *There is an isomorphism $\lambda^{(-)} : [0, 1]^{\mathbb{N}_{\geq 1}} \rightarrow \text{Nat}(\mathcal{M}, \mathcal{D}_{\leq 1})$ given by*

$$\lambda_X^b(f)(x) = \begin{cases} \frac{f(x)}{\sum_{x \in X} f(x)} \cdot b\left(\sum_{x \in X} f(x)\right) & \text{if } \sum_{x \in X} f(x) > 0, \\ 0 & \text{if } \sum_{x \in X} f(x) = 0. \end{cases}$$

Characterisation of $F_A \Rightarrow F_B$

We characterise $\text{Nat}(F_A, F_B)$

- $\mathcal{P}_f \Rightarrow \mathcal{P}_f$ and $\mathcal{M} \Rightarrow \mathcal{M}$
[Dahlqvist&Neves'18]

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(Non-gray entries are our new results)

Remark: $F_{(_)}$ is faithful but **not full**.

- In **CMon**, any morph $i: (\mathbb{N}, +, 0) \rightarrow (2, \vee, \perp)$ satisfies $i(n) = n \cdot i(1)$.
Thus **CMon** $((\mathbb{N}, +, 0), (2, \vee, \perp)) \cong 2$.
- But $\text{Nat}(F_{\mathbb{N}}, F_2) = \text{Nat}(\mathcal{M}, \mathcal{P}_f) \cong 2^{\mathbb{N}_{\geq 1}}$ holds.

Characterisation of $F_A \Rightarrow F_B$

We study a part of these two cases:

$F_A \setminus F_B$	$\mathcal{P}_f$	$\mathcal{M}$	$\mathcal{R}_{\geq 0}^+$	$\mathcal{R}_{\geq 0}^\times$
$\mathcal{P}_f$	2	1	1	2
$\mathcal{M}$	$2^{\mathbb{N}_{\geq 1}}$	$\mathbb{N}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$	$\mathbb{R}_{\geq 0}^{\mathbb{N}_{\geq 1}}$
$\mathcal{R}_{\geq 0}^+$	$2^{\mathbb{R}_{>0}}$	1	$\mathbb{R}_{\geq 0}^{\mathbb{R}_{>0}}$	?

1. When A is singly generated

- $\mathcal{P}_f \Rightarrow \mathcal{D}_{\leq 1}$ is unique
... no-go thm for MCs and NFAs
- $\mathcal{M} \Rightarrow \mathcal{R}_{\geq 0}^+$ is determined by $b: \mathbb{N}_{\geq 1} \rightarrow \mathbb{R}_{\geq 0}$
... prod. const. for MCs and MFAs

2. When $A = (\mathbb{R}_{\geq 0}, +, 0)$

- $\mathcal{D}_{\leq 1} \Rightarrow \mathcal{R}_{\geq 0}^+$ is determined by $b: [0,1] \rightarrow \mathbb{R}_{\geq 0}$
... prod. const. for MCs and MFAs

Characterisation of natural trans

When A is singly generated

Case 1: $A = (\perp \rightarrow a \rightarrow a^2 \rightarrow \dots)$

There exists an isomorphism $\lambda^{(-)}: \{b \in B^{\mathbb{N}} \mid b(0) = 0\} \rightarrow \text{Nat}(F_A, F_B)$:

$$\lambda_X^b(f)(x) := N(f(x)) \cdot b\left(\sum_{x \in X} N(f(x))\right).$$

- e.g. $\mathcal{M} \Rightarrow \mathcal{R}_{\geq 0}^+$ is determined by $b: \mathbb{N}_{\geq 1} \rightarrow \mathbb{R}_{\geq 0}$

Case 2: $A = (\perp \rightarrow a \rightarrow \dots \rightarrow a^{n-1})$

Let n be the least natural number such that $n > 1$ and $n \cdot a = a$. There exists an isomorphism $\lambda^{(-)}: \{c \in B \mid n \cdot c = c\}^{\{0, \dots, n-2\}} \rightarrow \text{Nat}(F_A, F_B)$ given by

$$\lambda_X^b(f)(x) := N(f(x)) \cdot b\left(\left[\sum_{x \in X} N(f(x))\right]\right),$$

where $[l]$ is the remainder of l modulo $n - 1$.

- e.g. $\mathcal{P}_f \Rightarrow \mathcal{R}_{\geq 0}^+$ is unique. $(\{c \mid c^{\mathbb{Z}} = c\}^0 \cong 1)$

When A is singly generated

We can extend the result for natural trans between subfunctors

Prop.

- No natural trans $\mathcal{P}_f \Rightarrow \mathcal{D}$ exists.
- There exists a unique $\mathcal{P}_f \Rightarrow \mathcal{D}_{\leq 1}$.
- No natural trans $\mathcal{M} \Rightarrow \mathcal{D}$ exists.
- There is an isom $\lambda^{(-)}: [0,1]^{\mathbb{N}_{\geq 1}} \rightarrow \text{Nat}(\mathcal{M}, \mathcal{D}_{\leq 1})$ defined by

$\mathcal{P}_f \Rightarrow \mathcal{R}_{\geq 0}^+$ is unique, and is Δ_0 .
 Δ_0 in $\mathcal{D}_{\leq 1}X$ but not in $\mathcal{D}X$

$$\lambda_X^b(f)(x) = \begin{cases} \frac{f(x)}{\sum_{x \in X} f(x)} \cdot b\left(\sum_{x \in X} f(x)\right) & \text{if } \sum_{x \in X} f(x) > 0, \\ 0 & \text{if } \sum_{x \in X} f(x) = 0. \end{cases}$$

Conclusion

In the framework of coalgebraic product construction,

$$\begin{array}{ccc} \text{System } X \xrightarrow{c} F_S X, & & \text{Product } X \times Y \xrightarrow{c \otimes_\lambda d} F_{S \otimes R}(X \times Y) \\ \text{Req. } Y \xrightarrow{d} F_R Y & \mapsto & \text{s.t. } \llbracket c \otimes_\lambda d \rrbracket \text{ represents the target problem} \end{array}$$

$$\lambda: \times \circ (F_S \times F_R) \Rightarrow F_{S \otimes R} \circ \times$$

(well-behaved for semantics)

We show that

- There is no product construction for MCs and NFAs
→ first formal proof of no-go thm for product constructions
- There is a unique product construction for MCs and MFAs
→ new instance

The proof relies on our characterisation of natural trans. $F_A \Rightarrow F_B$.