

From Coalgebraic Determinization to Belief Construction for Partial Observability

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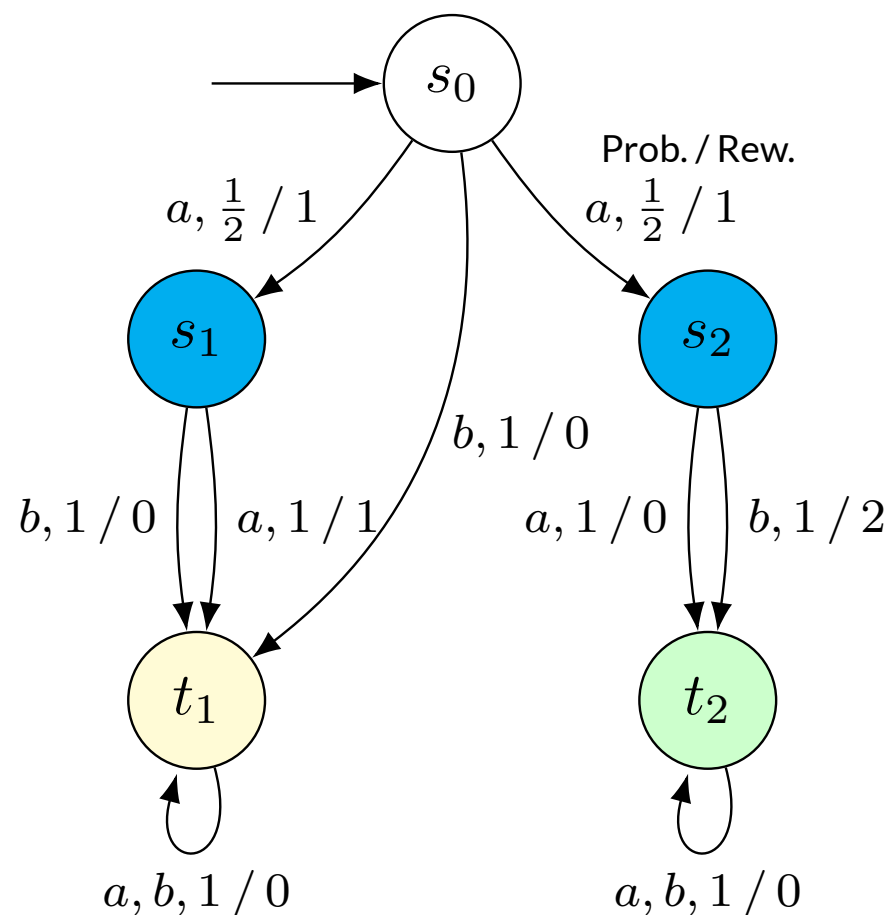
Background

Partially Observable MDP (POMDP)

a POMDP $S \rightarrow (\mathcal{D}S \times \mathbb{R}_{\geq 0})^A \times \mathcal{O}$

= an MDP $(\delta: S \times A \rightarrow \mathcal{D}S, \text{rew}: S \times A \rightarrow \mathbb{R}_{\geq 0})$

+ an observation map $\text{obs}: S \rightarrow \mathcal{O}$



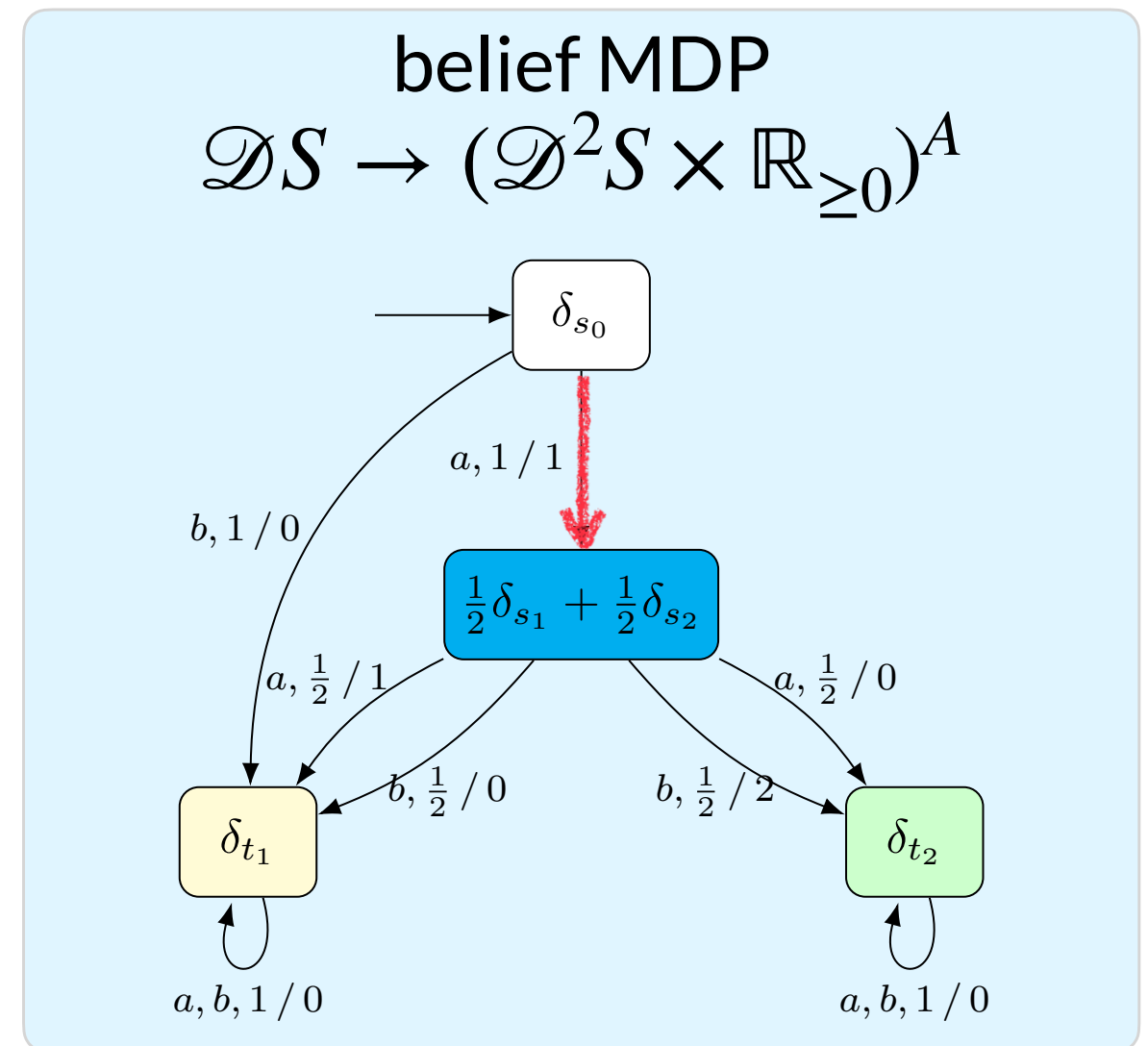
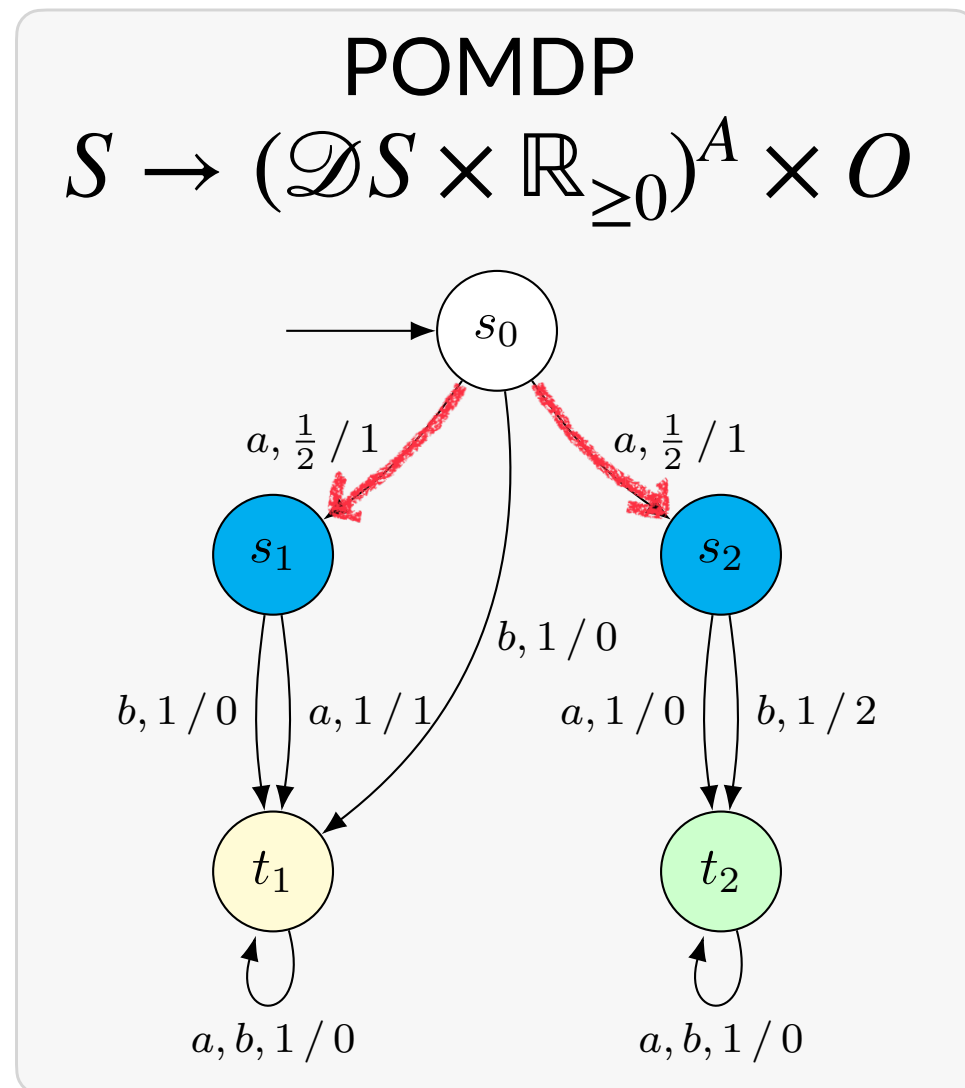
Objective: $\max_{\pi} \sum_{p \in \text{Path}_{\pi}(s)} \text{Pr}(p) \cdot \text{Rew}(p)$

maximal expected reward
under observation-based schedulers
 $\pi: \mathcal{O} \times (A \times \mathcal{O})^* \rightarrow A$.

= LFP of the Bellman operator
if it is **fully observable**

Background

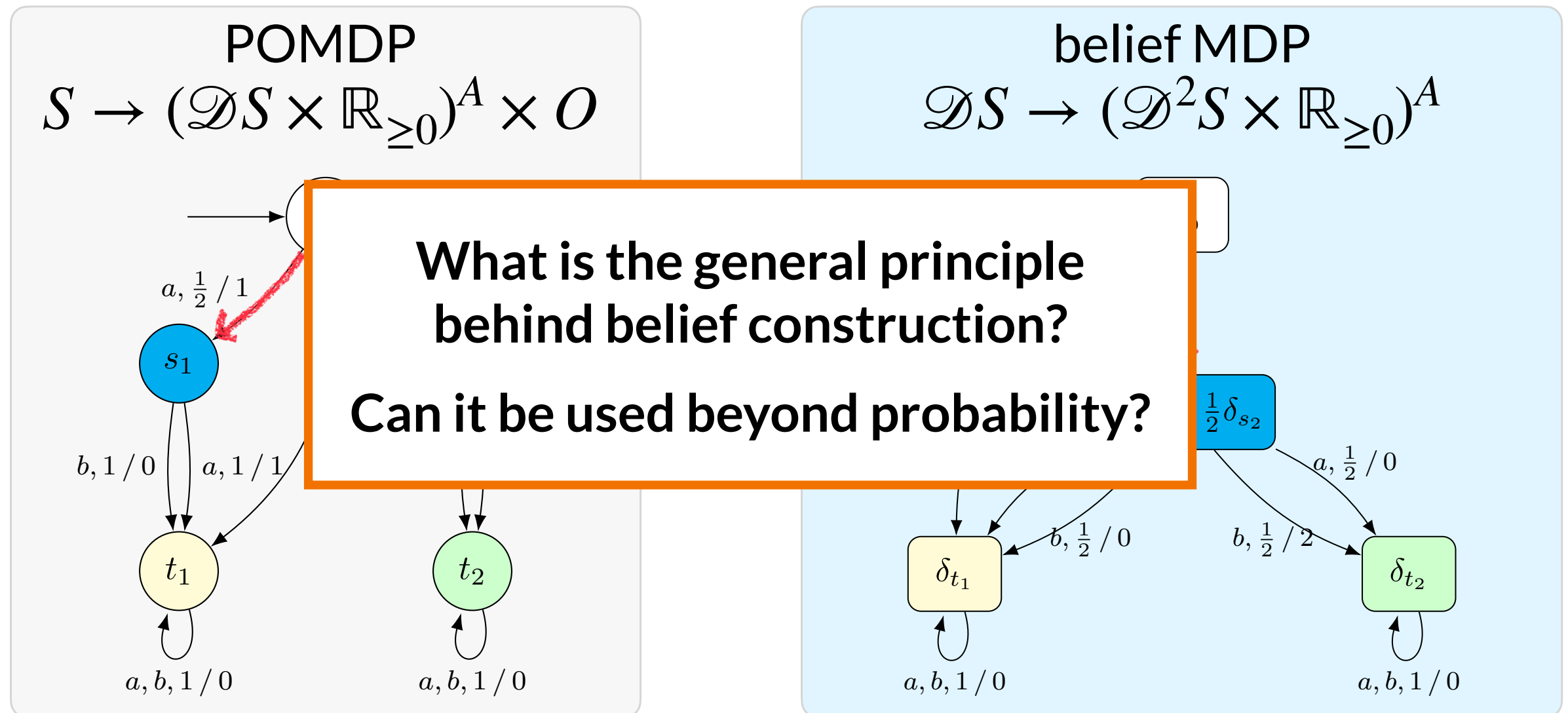
Belief Construction



$\text{MaxERew}(\text{POMDP}) = \text{MaxERew}(\text{belief MDP})$
 $= \text{LFP of the Bellman operator}$

Background

Belief Construction



$$\begin{aligned} \text{MaxERew}(\text{POMDP}) &= \text{MaxERew}(\text{belief MDP}) \\ &= \text{LFP of the Bellman operator} \end{aligned}$$

Our Contribution

Coalgebraic Belief Construction

A generic belief construction for partially observable systems

partially observable FT-coalgebra

$$I \xrightarrow{i} S \xrightarrow{c=\langle \delta, \text{obs} \rangle} FTS \times O$$

where F : one-step behaviour
 T : branching type

Bel
 $\mapsto$

belief coalgebra

$$I \rightarrow B_{\text{obs}} \rightarrow FTB_{\text{obs}} \times O$$

where B_{obs} : space of
(observation-consistent) belief states

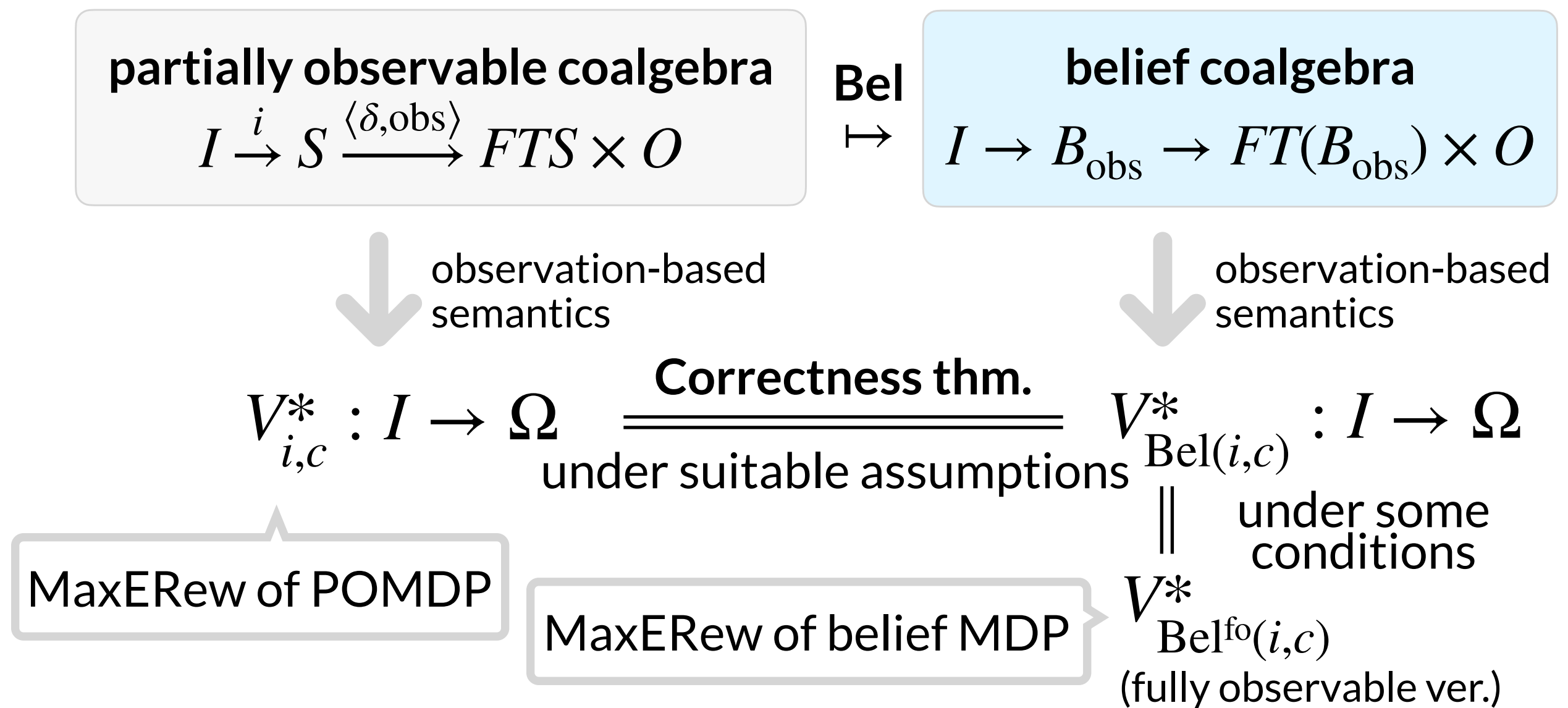
Technical ingredients:

- Coalgebraic determinization [SilvaBBR, LMCS'13]
- Belief decomposition $\alpha: Td \Rightarrow Td\bar{T}$,
which generalises conditioning by observations

We lift monad $T: \mathbb{C} \rightarrow \mathbb{C}$
to $\bar{T}: \mathbb{C}/O \rightarrow \mathbb{C}/O$

Our Contribution

Correctness of the construction



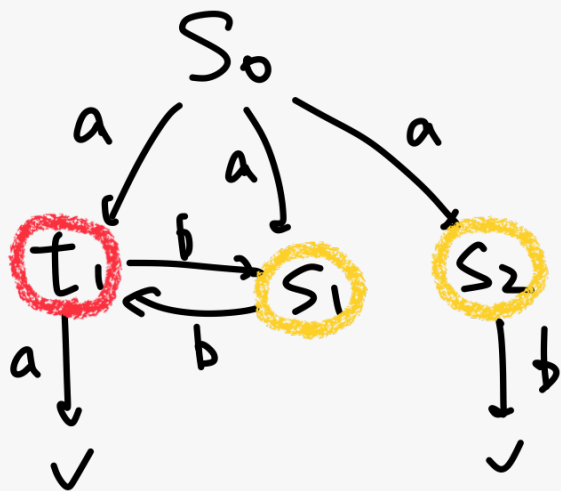
Our contribution

Instances beyond Probability | Nondeterminism

PO nondeterministic transition systems

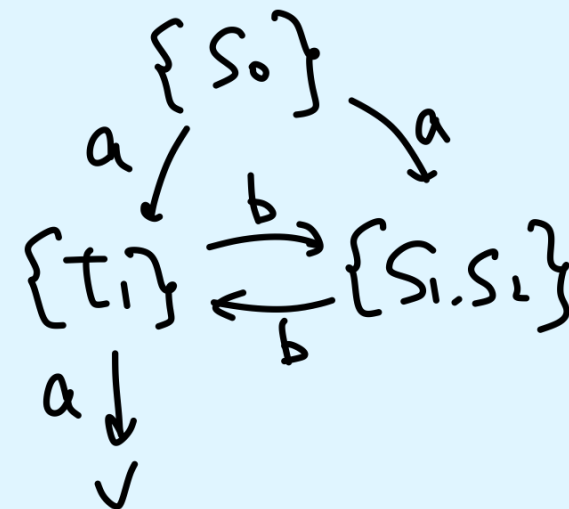
$$T := \mathcal{P}_{\neq \emptyset}, F := (_ + \{\checkmark\})^A$$

$$c: S \rightarrow (\mathcal{P}_{\neq \emptyset} S + \{\checkmark\})^A \times O$$



$\mapsto$

$$c^{\text{Bel}}: B_{\text{obs}} \rightarrow (\mathcal{P}_{\neq \emptyset}(B_{\text{obs}}) + \{\checkmark\})^A$$



It preserves an existence of an observation-based scheduler making every path eventually reach $\checkmark$.

Our contribution

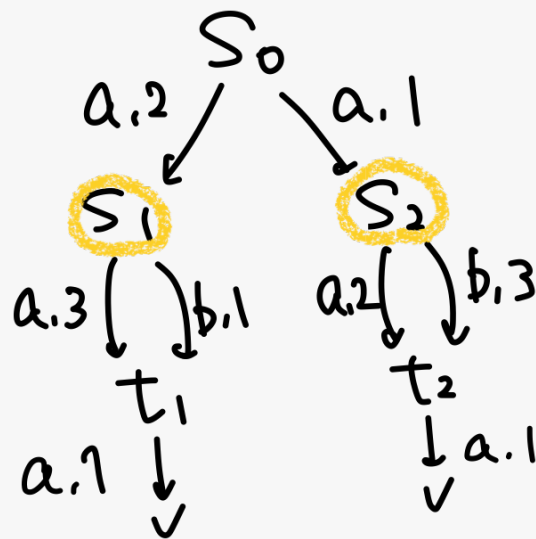
Instances beyond Probability | Weights

New belief construction

PO weighted transition systems

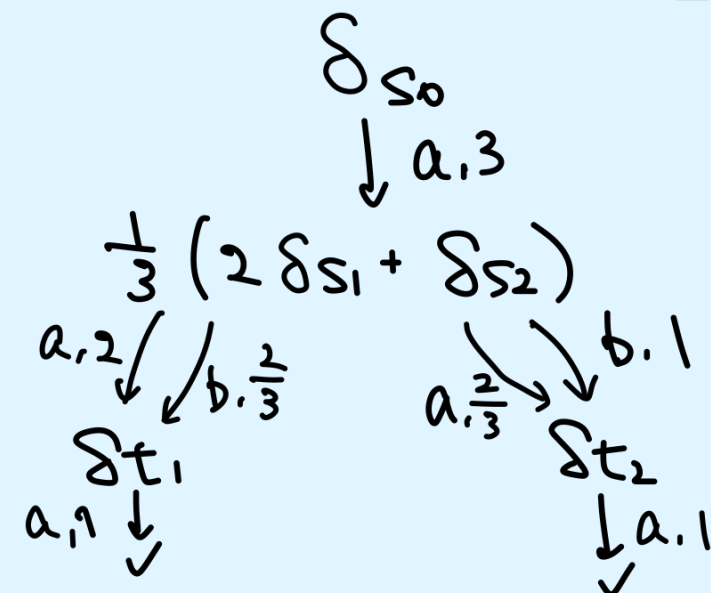
$T := \mathcal{R}, F := (_ \times \mathbb{R}_{\geq 0})^A$ where $\mathcal{R}(S) = \{\nu: S \rightarrow \mathbb{R}_{\geq 0} \mid \nu \text{ is finitely supported}\}$

$$c: S \rightarrow (\mathcal{R}S \times \mathbb{R}_{\geq 0})^A \times O$$



$\mapsto$

$$c^{\text{Bel}}: B_{\text{obs}} \rightarrow (\mathcal{R}(B_{\text{obs}}) \times \mathbb{R}_{\geq 0})^A \times O$$

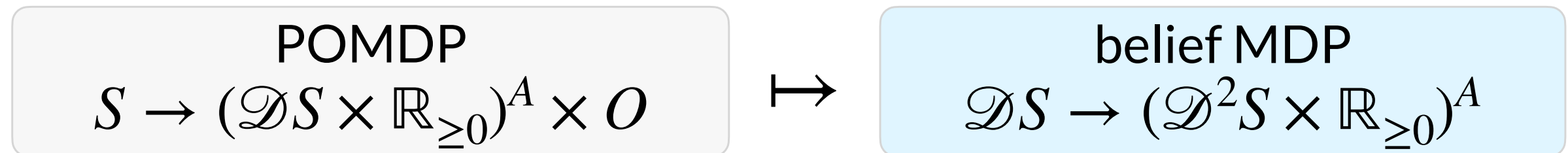


It preserves the maximal total weight of terminating paths under observation-based schedulers.

Coalgebraic Belief Construction
= coalgebraic determinization
+ decomposition by observations

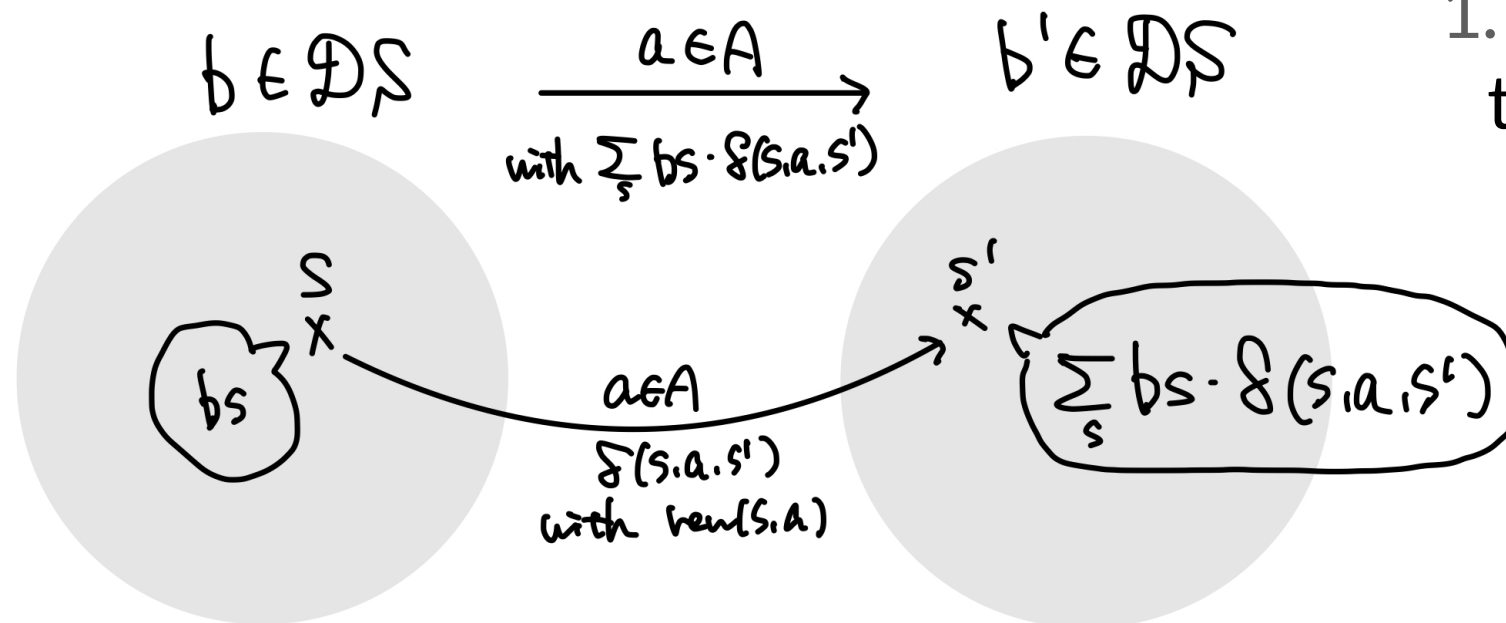
Coalgebraic belief construction

Belief Construction for POMDPs



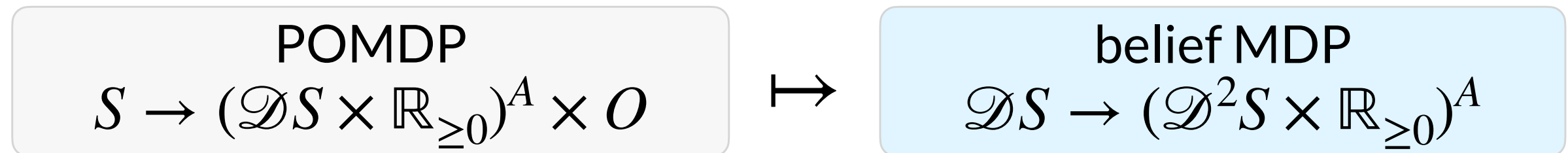
Construction of a transition of the belief MDP:

1. Lift transitions to the level of beliefs



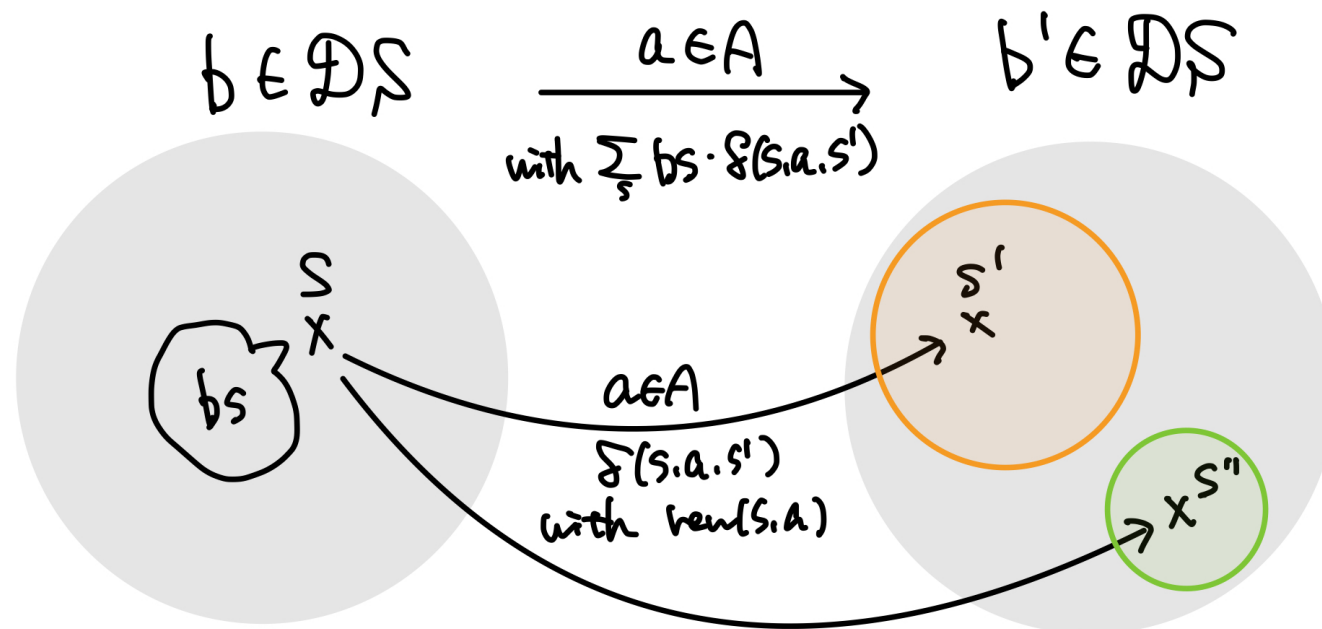
Coalgebraic belief construction

Belief Construction for POMDPs



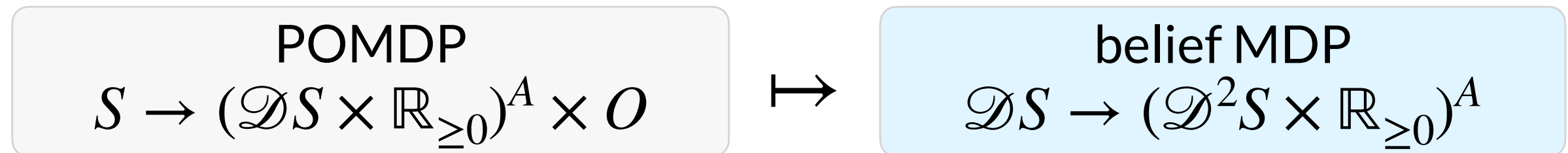
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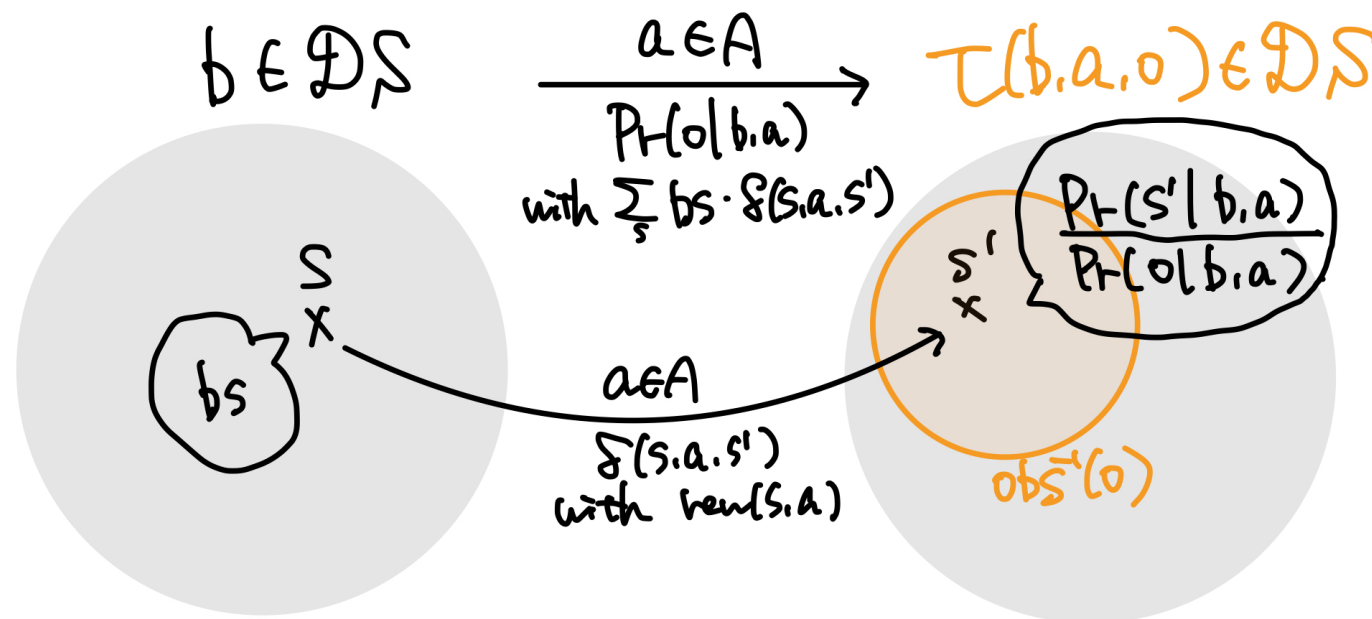


Coalgebraic belief construction

Belief Construction for POMDPs



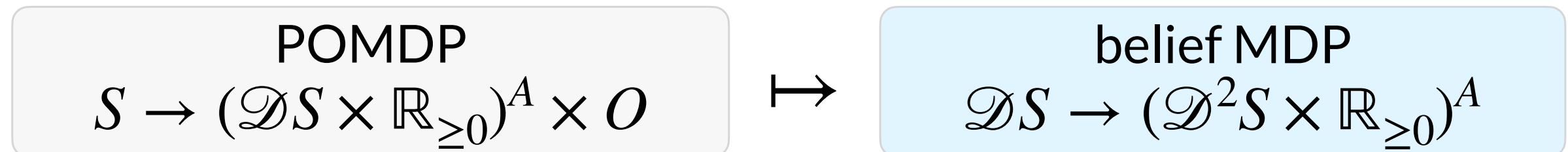
Construction of a transition of the belief MDP:



1. Lift transitions to the level of beliefs
2. Split and condition by observations

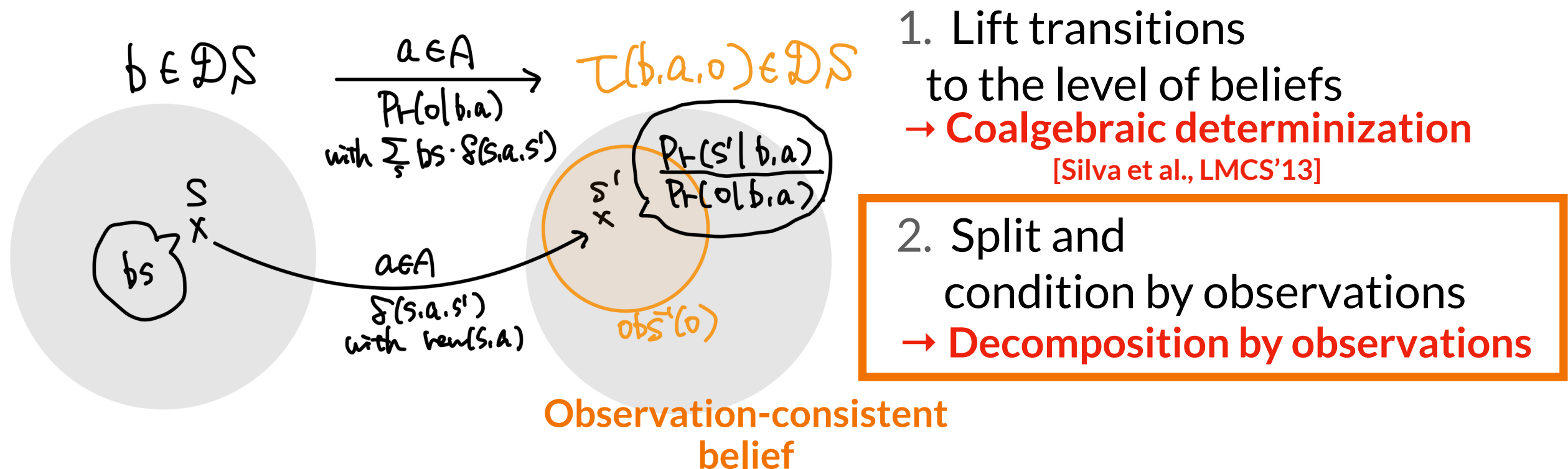
Coalgebraic belief construction

Belief Construction for POMDPs



Partially Observable Coalgebra $S \rightarrow FTS \times O$
 for a functor F , a monad T (e.g. $F = (- \times \mathbb{R}_{\geq 0})^A, T = \mathcal{D}$)

Construction of a transition of the belief MDP:



Coalgebraic belief construction

Lifting a Monad to the Slice Category

a monad $T : \mathbb{C} \longrightarrow \mathbb{C}$
 states $S \mapsto$ beliefs TS

Lift along
 $d : \mathbb{C}/O \rightarrow \mathbb{C}$

a monad $\bar{T} : \mathbb{C}/O \longrightarrow \mathbb{C}/O$
 $(S \xrightarrow{\text{obs}} O) \mapsto (B_{\text{obs}} \xrightarrow{\bar{T}(\text{obs})} O)$

states
with observations

observation-consistent
beliefs

$$\begin{array}{ccc}
 B_{\text{obs}} & \xrightarrow{\bar{T}(\text{obs})} & O \\
 \downarrow \iota_{\text{obs}} & & \downarrow \eta_O \\
 TS & \xrightarrow{T(\text{obs})} & TO
 \end{array}$$

e.g. $T := \mathcal{D} : \mathbf{Set} \rightarrow \mathbf{Set}$,

$$B_{\text{obs}} = \{b \in \mathcal{D}S \mid \exists o \in O. \text{supp}(b) \subseteq \text{obs}^{-1}(o)\},$$

$$\bar{T}(\text{obs})(b) = \text{the unique } o \text{ s.t. } \text{supp}(b) \subseteq \text{obs}^{-1}(o)$$

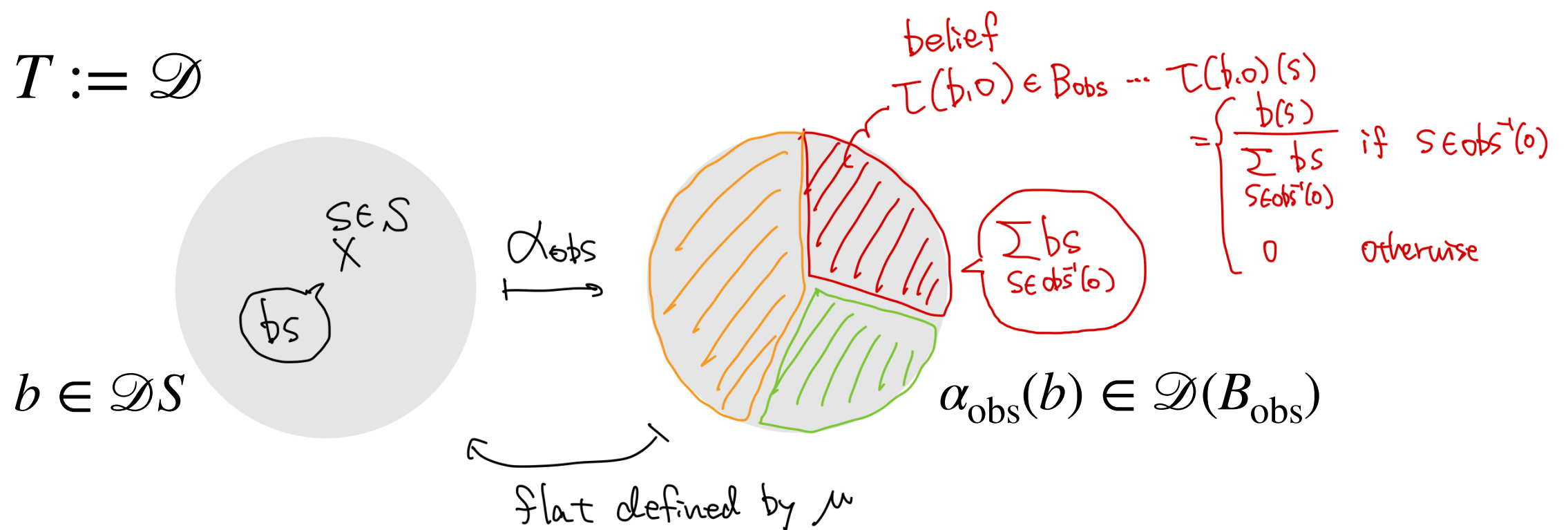
Coalgebraic belief construction

Belief Decomposition

Def. A belief decomposition is a natural trans. $\alpha: Td \Rightarrow Td\bar{T}$
 s.t. $\text{flat} \circ \alpha = \text{id}$ where $\{\text{flat}_f: T(B_f) \xrightarrow{Tl} T^2X \xrightarrow{\mu} TX\}_{f:X \rightarrow O}$.

$\alpha_{\text{obs}}: TS \rightarrow T(B_{\text{obs}})$ splits beliefs $b \in TS$ by observations.

e.g. $T := \mathcal{D}$



Coalgebraic Belief Construction

Def. Let $\lambda: TF \Rightarrow FT$ be a distributive law,
 $\alpha: Td \Rightarrow Td\bar{T}$ be a belief decomposition.

Given $c: S \xrightarrow{\langle \delta, \text{obs} \rangle} FTS \times O$,

$$c^{\text{Bel}} := \left(\underbrace{B_{\text{obs}} \xrightarrow{\iota_{\text{obs}}} TS}_{\text{forget the obs. index}} \underbrace{\xrightarrow{\text{Det}_{\lambda}(c)} FTS}_{\text{Coalgebraic determinization}} \underbrace{\xrightarrow{F(\alpha_{\text{obs}})} FT(B_{\text{obs}})}_{\text{decomposition by observations.}} \right)$$

$$\begin{array}{ccc} B_{\text{obs}} & \xrightarrow{\bar{T}(\text{obs})} & O \\ \downarrow \iota_{\text{obs}} & & \downarrow \eta_O \\ TS & \xrightarrow{T(\text{obs})} & TO \end{array}$$

It gives a functor

pointed PO coalgebra

$$I \xrightarrow{i} S \xrightarrow{\langle \delta, \text{obs} \rangle} FTS \times O$$

Bel
 $\mapsto$

pointed PO coalgebra

$$I \rightarrow B_{\text{obs}} \xrightarrow{\langle c^{\text{Bel}}, \bar{T}(\text{obs}) \rangle} FT(B_{\text{obs}}) \times O$$

Conclusions

Coalgebraic belief construction

- A generic, semantics-preserving belief construction
- For a PO coalgebra $c: S \rightarrow FTS \times O$,

$$c^{\text{Bel}} := \left(\underbrace{B_{\text{obs}} \xrightarrow{l_{\text{obs}}} TS}_{\text{forget the obs. index}} \xrightarrow{\text{Coalgebraic determinization } \text{Det}_\lambda(c)} \underbrace{FTS \xrightarrow{F(\alpha_{\text{obs}})} FT(B_{\text{obs}})}_{\text{decomposition by observations.}} \right)$$

- Instances for: POMDPs, PO nondet. systems, PO weighted systems
New belief construction

Ongoing work: Generalise the correctness thm using coalgebraic trace semantics and effectful schedulers