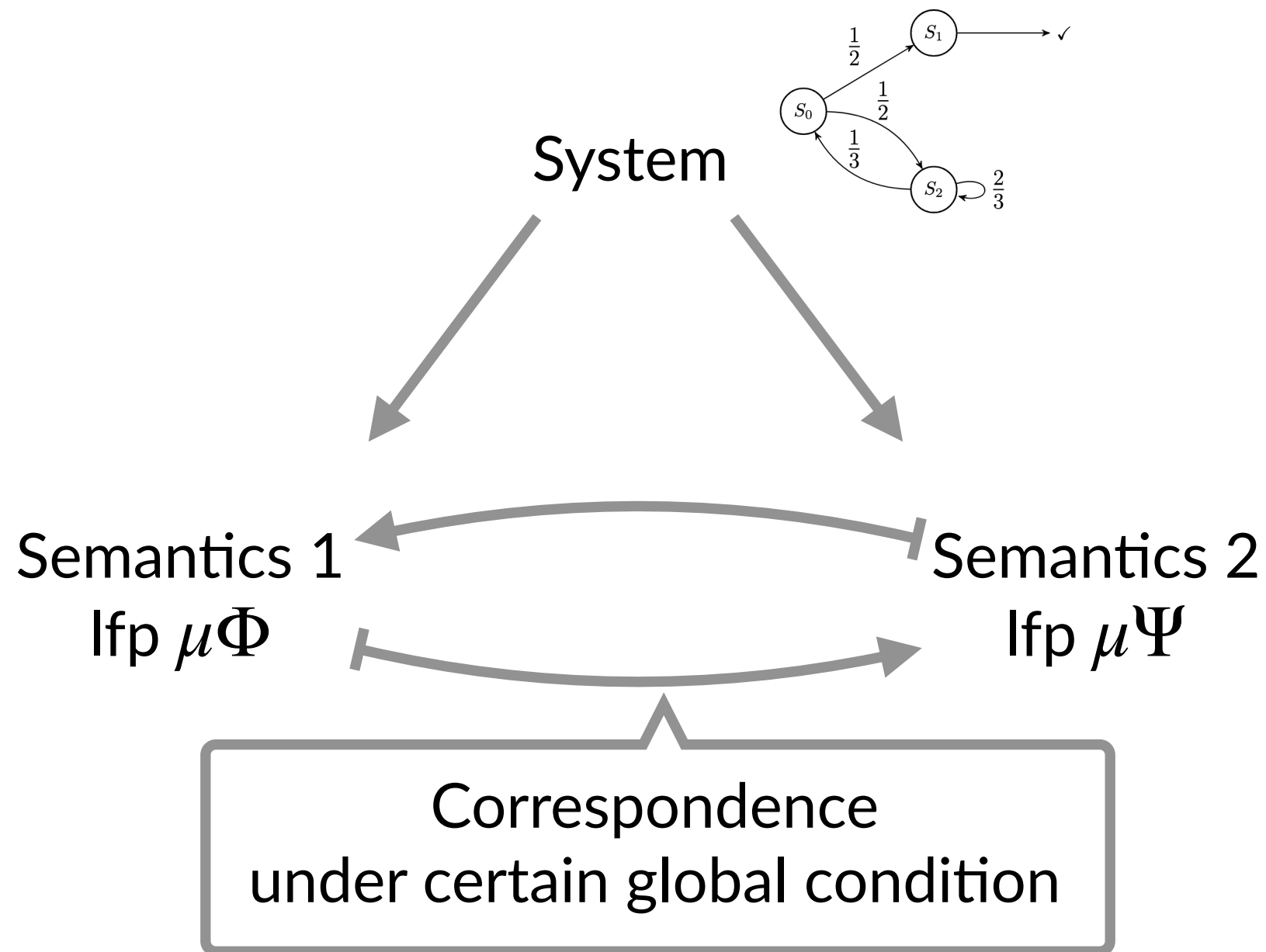


Initial Algebra Correspondence under Reachability Conditions

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Main topic

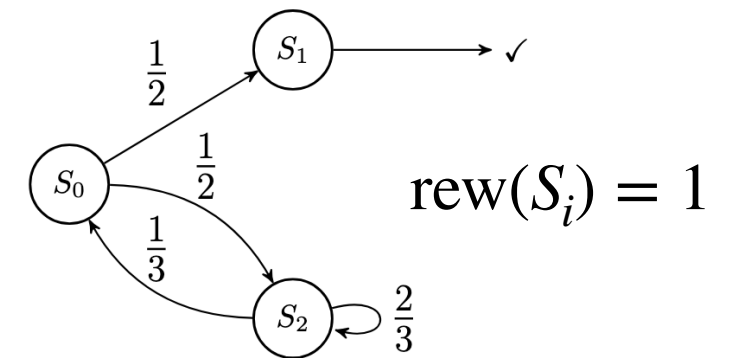
Correspondence of two fixed point semantics



Example

Total & Partial expected reward

Markov Chain $\langle P, \text{rew} \rangle : S \rightarrow \mathcal{D}(S + \{\checkmark\}) \times \mathbb{N}$



- Total expected reward: erew for all paths (μ^Ψ)

$$\Psi : [S, [0, \infty]] \rightarrow [S, [0, \infty]]$$

$$k \mapsto (s \mapsto \text{rew}(s) + \sum_{s'} P(s, s') \cdot k(s'))$$

- Partial expected reward: erew for paths reaching $\checkmark$ (μ^Φ)

$$\Phi : [S, [0, 1] \times [0, \infty]] \rightarrow [S, [0, 1] \times [0, \infty]]$$

$$k \mapsto (s \mapsto (p, r))$$

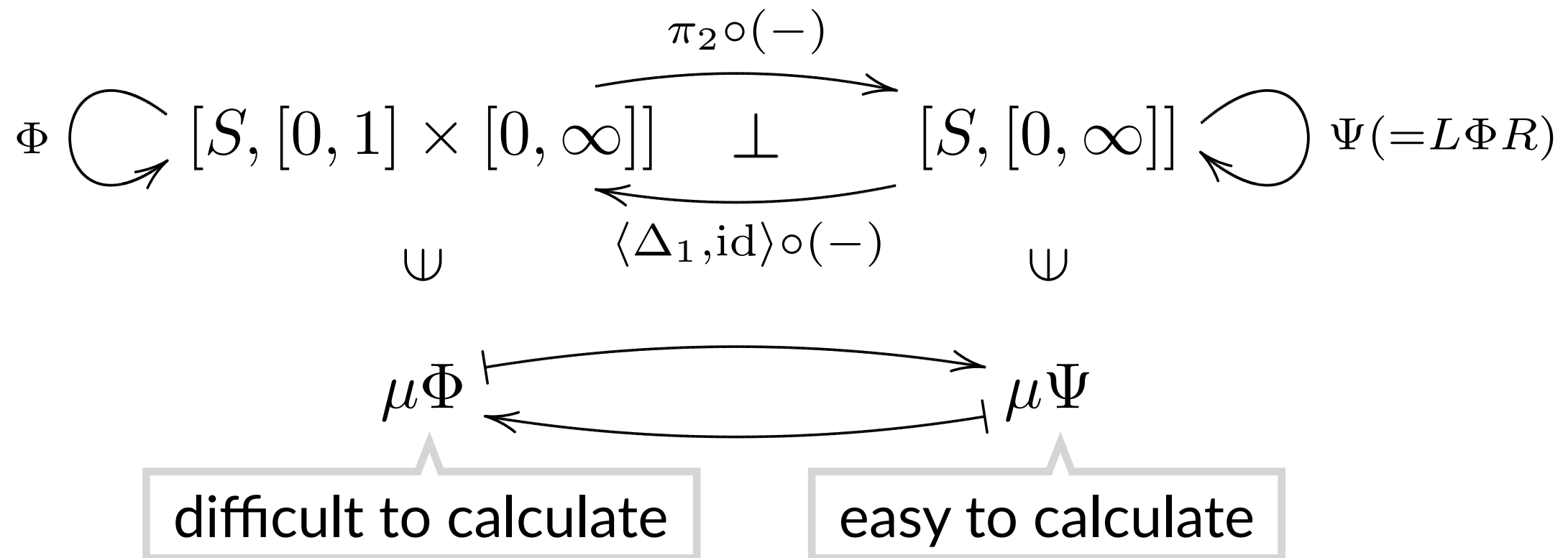
where $p := P(s, \checkmark) + \sum_{s' \in S} P(s, s') \cdot \pi_1 k(s')$

$$r := \text{rew}(s) \cdot P(s, \checkmark) + \sum_{s' \in S} P(s, s') \cdot (\pi_2 k(s') + \text{rew}(s) \cdot \pi_1 k(s'))$$

Example

Correspondence of expected rewards

Prop. If a MC is almost surely reachable ($\forall s \in S. \Pr(s, \checkmark) = 1$), then total erew $\mu\Psi$ and partial erew $\mu\Phi$ correspond.

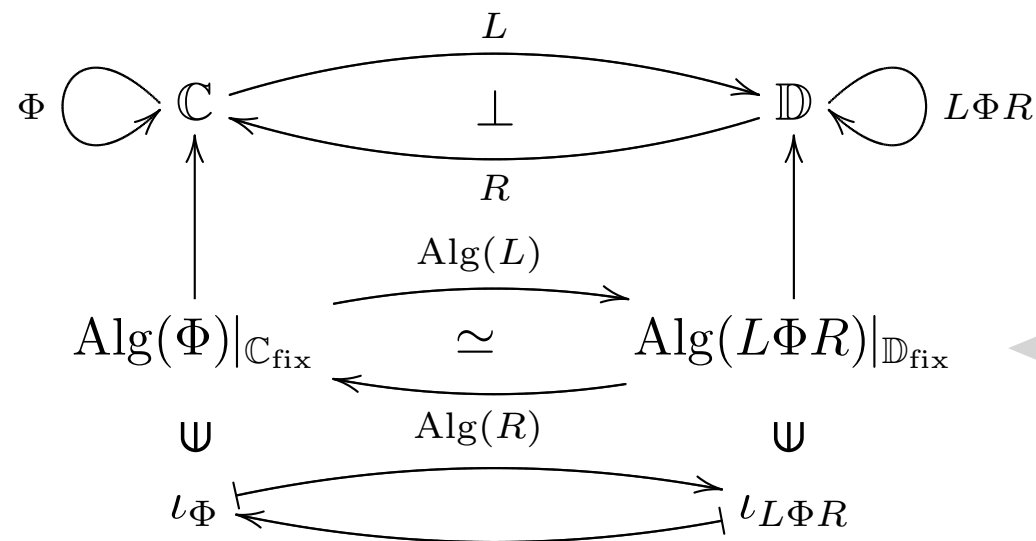


How does such correspondence under global condition occur?

How can we obtain such condition?

Framework for initial algebra correspondence

1. Initial algebra ι_Φ and $\iota_{L\Phi R}$ correspond via adjoints



adjoint equivalence
btw. subcategories
of algebras

- Global condition for this correspondence
- e.g. almost surely reachability of MC, MDP, resource-bounded reachability

2. Extension for $\Phi \hookrightarrow \mathbb{C} \begin{matrix} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{matrix} \mathbb{D} \hookrightarrow \Psi$ with $\rho: L\Phi R \Rightarrow \Psi$

- e.g. unambiguity of NFA

Comparison to related works

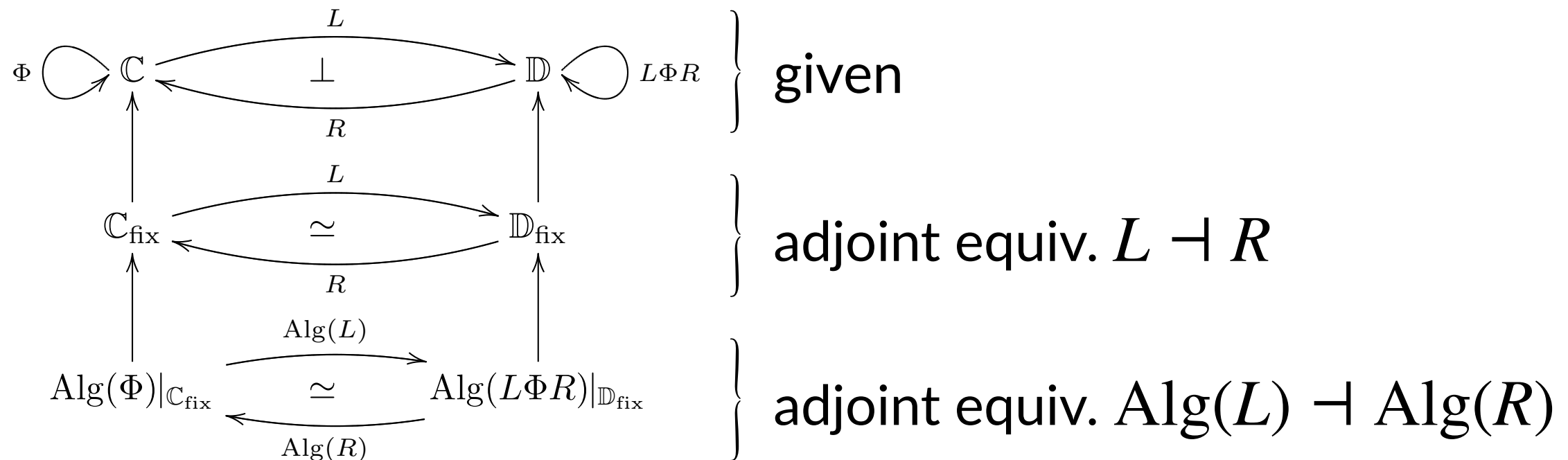
- In lattice theory:

completeness of abstract domains [Cousot&Cousot, POPL'79, Giacobazzi+, J. ACM'00]	our conditions
local condition $\Psi L = L\Phi$ to ensure $L(\mu\Phi) = \mu\Psi$	global condition to ensure $L(\mu\Phi) = \mu\Psi$ and $\mu\Phi = R(\mu\Psi)$
It induces initial chain corresp.	chains may not correspond

- In category theory: Preservation of initial algebras/final coalgebras
[Hermida&Jacobs, Inf. Comput. '98], [SprungerKDH, J. Log. Comput.'21]
 - Holds in general
 - Ours: Holds only under global condition

Outline

1. Adjoint equivalences

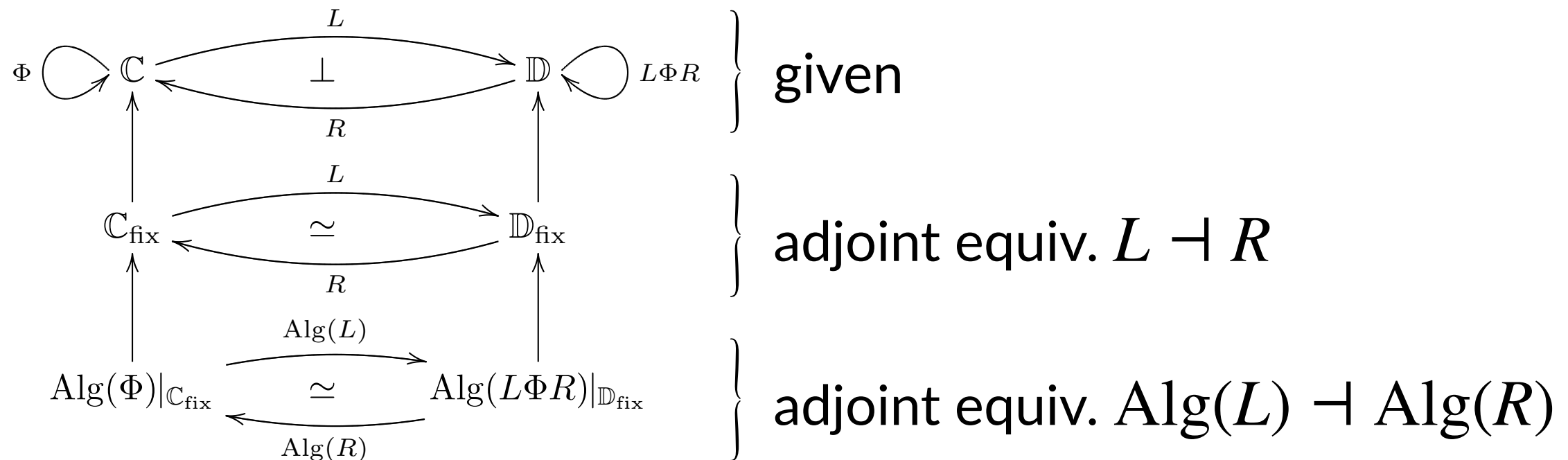


2. Initial algebra corresp. for Φ and $L\Phi R$

3. Initial algebra corresp. for Φ and Ψ with a step

Outline

1. Adjoint equivalences



2. Initial algebra corresp. for Φ and $L\Phi R$

3. Initial algebra corresp. for Φ and Ψ with a step

Adjoint equivalence

Adjoints mapping initial algebras

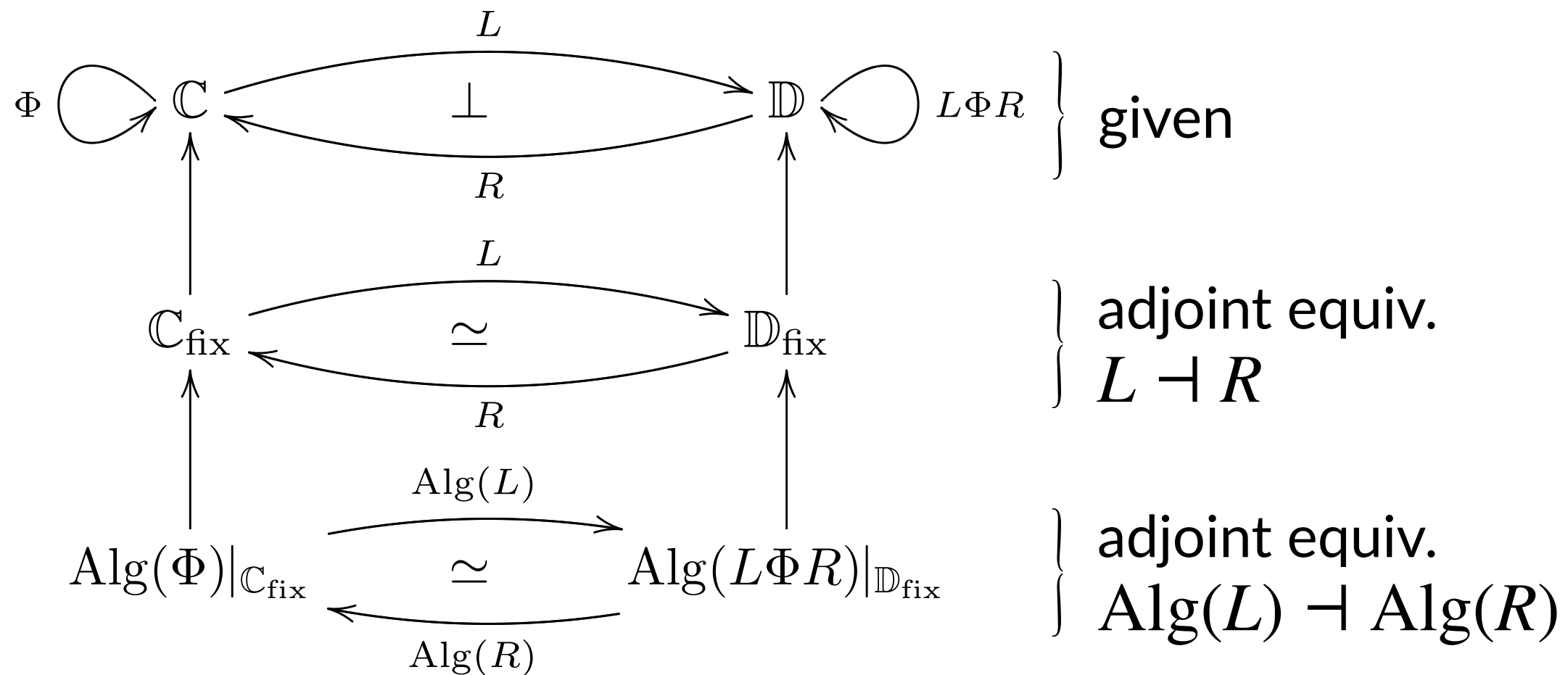
$$\begin{array}{c} \Phi \circlearrowright \mathbb{C} \begin{array}{c} \xrightarrow{L} \mathbb{D} \circlearrowleft L\Phi R \\ \xleftarrow{R} \end{array} \\ \perp \\ \text{Alg}(\Phi) \ni \iota_\Phi \begin{array}{c} \xrightarrow{?} \iota_{L\Phi R} \in \text{Alg}(L\Phi R) \\ \xleftarrow{\quad} \end{array} \end{array}$$

Which functors do we use to map initial algebras $\iota_\Phi, \iota_{L\Phi R}$?

- $\text{Alg}(R): \text{Alg}(L\Phi R) \rightarrow \text{Alg}(\Phi)$ always exists
by $\eta: \Phi R \Rightarrow R(L\Phi R)$ [Thm 2.14, Hermida&Jacobs'98].
- $\text{Alg}(L): \text{Alg}(\Phi) \rightarrow \text{Alg}(L\Phi R)$ may not exist in general
 - We derive adjoints between **subcategories of** $\text{Alg}(\Phi), \text{Alg}(L\Phi R)$
 - It forms an **adjoint equivalence**, i.e. adjunction with isom η, ϵ

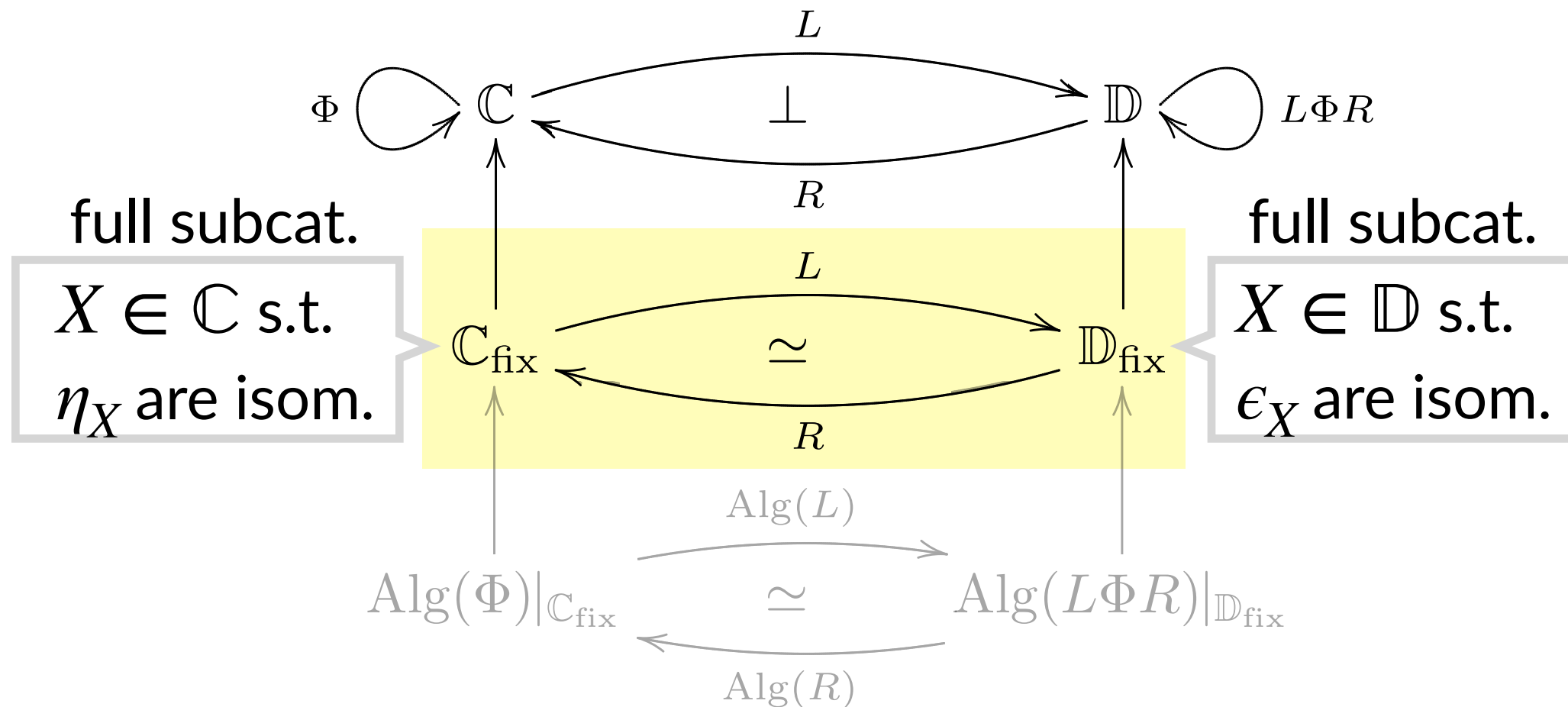
Adjoint equivalence

Adjoint equivalence btw. subcategories



Adjoint equivalence

Adjoint equivalence btw. subcategories



Prop (folklore). Any $(L, R; \eta, \epsilon): \mathbb{C} \rightarrow \mathbb{D}$ restricts to an adjoint equivalence $(L, R; \eta, \epsilon): \mathbb{C}_{\text{fix}} \rightarrow \mathbb{D}_{\text{fix}}$, i.e. $L \dashv R$ and both η and ϵ are isomorphisms.

Functor btw subcategories of algebras

Def. Let $\Phi: \mathbb{C} \rightarrow \mathbb{C}$. $\text{Alg}(\Phi)$ is the category of Φ -algebras.

For subcat. $\mathbb{C}' \hookrightarrow \mathbb{C}$, define $\text{Alg}(\Phi)|_{\mathbb{C}'}$ by $\text{Alg}(\Phi)|_{\mathbb{C}'} \longrightarrow \text{Alg}(\Phi)$

$$\begin{array}{ccc} \text{Alg}(\Phi)|_{\mathbb{C}'} & \longrightarrow & \text{Alg}(\Phi) \\ \downarrow & & \downarrow \\ \mathbb{C}' & \hookrightarrow & \mathbb{C} \end{array}$$

Def. Let $\Phi: \mathbb{C} \rightarrow \mathbb{C}$, $\Psi: \mathbb{D} \rightarrow \mathbb{D}$, $\mathbb{C}' \hookrightarrow \mathbb{C}$, $\mathbb{D}' \hookrightarrow \mathbb{D}$.

Let $L: \mathbb{C} \rightarrow \mathbb{D}$ that can be restricted to $L: \mathbb{C}' \rightarrow \mathbb{D}'$.

$$\begin{array}{ccc} \mathbb{C}' & \xrightarrow{\Phi} & \mathbb{C} \\ L \downarrow & \nearrow \alpha & \downarrow L \\ \mathbb{D}' & \xrightarrow{\Psi} & \mathbb{D} \end{array}$$

induces

$$\text{Alg}(L)_\alpha: \text{Alg}(\Phi)|_{\mathbb{C}'} \rightarrow \text{Alg}(\Psi)|_{\mathbb{D}'}$$

$$(X, a: \Phi X \rightarrow X) \mapsto (LX, La \circ \alpha_X)$$

$$f \mapsto Lf$$

Adjoint equivalence $\text{Alg}(L) \dashv \text{Alg}(R)$

Thm (Extension of [Hermida&Jacobs, Inf. Comput.'98, Thm. 2.14]).

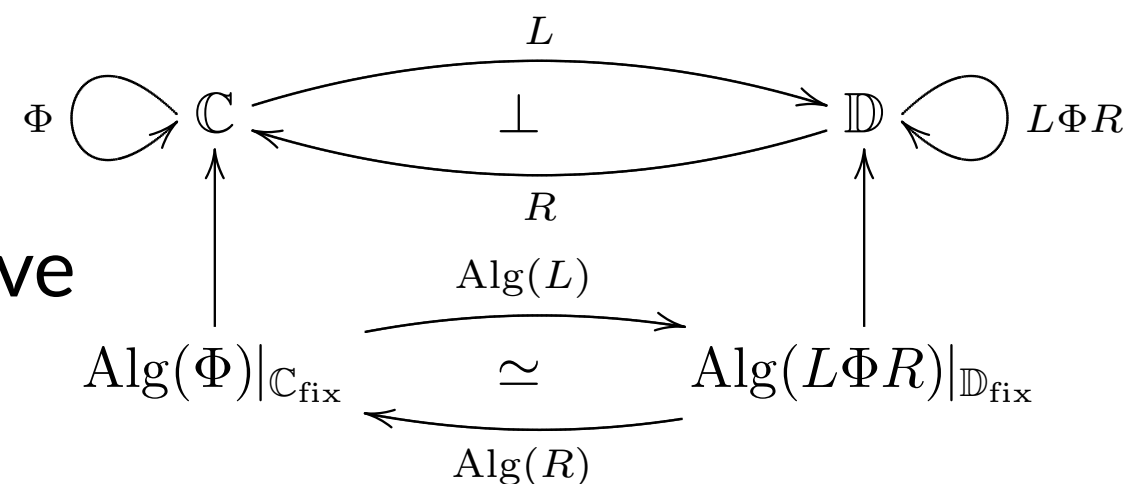
Let $\Phi: \mathbb{C} \rightarrow \mathbb{C}$, $\Psi: \mathbb{D} \rightarrow \mathbb{D}$, $\mathbb{C}' \hookrightarrow \mathbb{C}$, $\mathbb{D}' \hookrightarrow \mathbb{D}$.

$L \dashv R: \mathbb{D} \rightarrow \mathbb{C}$ (w/ restriction cond.) and $\begin{array}{ccc} \mathbb{C}' & \xrightarrow{\Phi} & \mathbb{C} \\ L \downarrow & \alpha \nearrow & \downarrow L \\ \mathbb{D}' & \xrightarrow{\Psi} & \mathbb{D} \end{array}$ imply

$\text{Alg}(L)_\alpha \dashv \text{Alg}(R)_{R\Psi_\epsilon \circ \bar{\alpha}R}: \text{Alg}(\Psi)|_{\mathbb{D}'} \rightarrow \text{Alg}(\Phi)|_{\mathbb{C}'}$.

When $\mathbb{C}' = \mathbb{C}_{\text{fix}}$ and $\mathbb{D}' = \mathbb{D}_{\text{fix}}$, it is an adjoint equivalence.

For $\Phi: \mathbb{C} \rightarrow \mathbb{C}$ and $L \dashv R: \mathbb{D} \rightarrow \mathbb{C}$,
by $\Psi := L\Phi R$ and $\alpha := L\Phi\eta^{-1}$, we have



Outline

1. Equivalence of subcategories of algebras

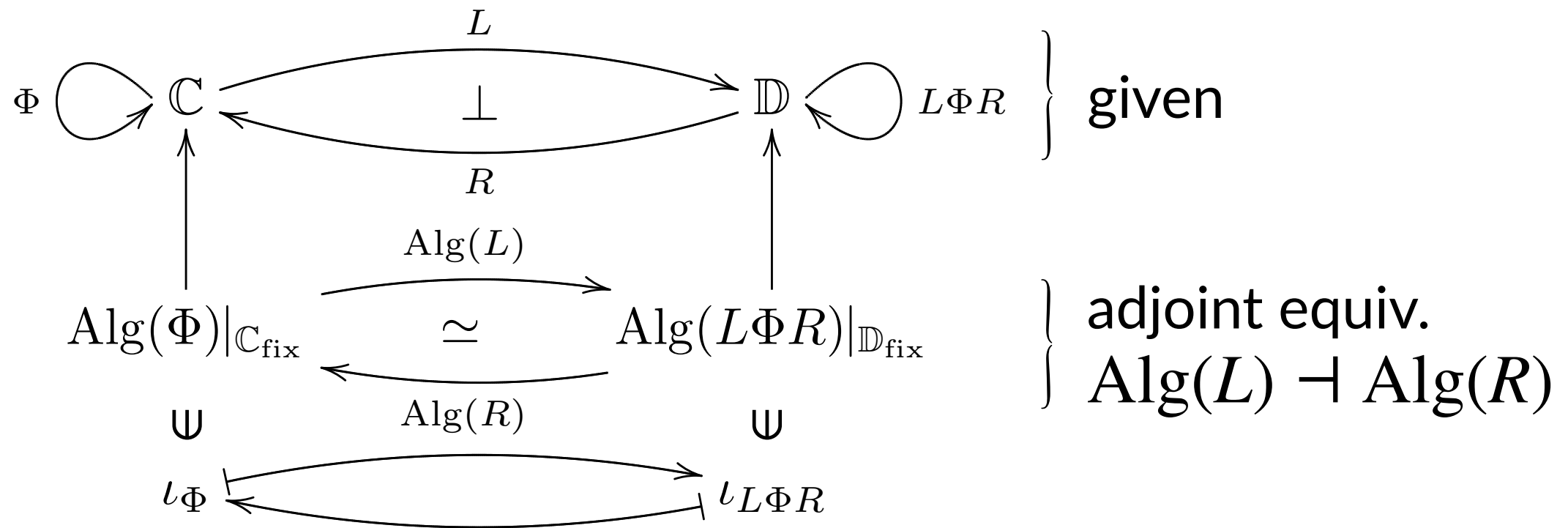


2. Initial algebra corresp. for Φ and $L\Phi R$

3. Initial algebra corresp. for Φ and Ψ with a step

Initial algebra corresp. for Φ and $L\Phi R$

Initial algebra correspondence



Def. Initial algebra corresp. for Φ and $L\Phi R$ holds

if ι_Φ is initial in $\text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}}$ and $\iota_{L\Phi R}$ is initial in $\text{Alg}(L\Phi R)|_{\mathbb{D}_{\text{fix}}}$.

(iff $\iota_\Phi \in \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}}$ and $\iota_{L\Phi R} \in \text{Alg}(L\Phi R)|_{\mathbb{D}_{\text{fix}}}$)

Initial algebra corresp. for Φ and $L\Phi R$

Global reachability condition

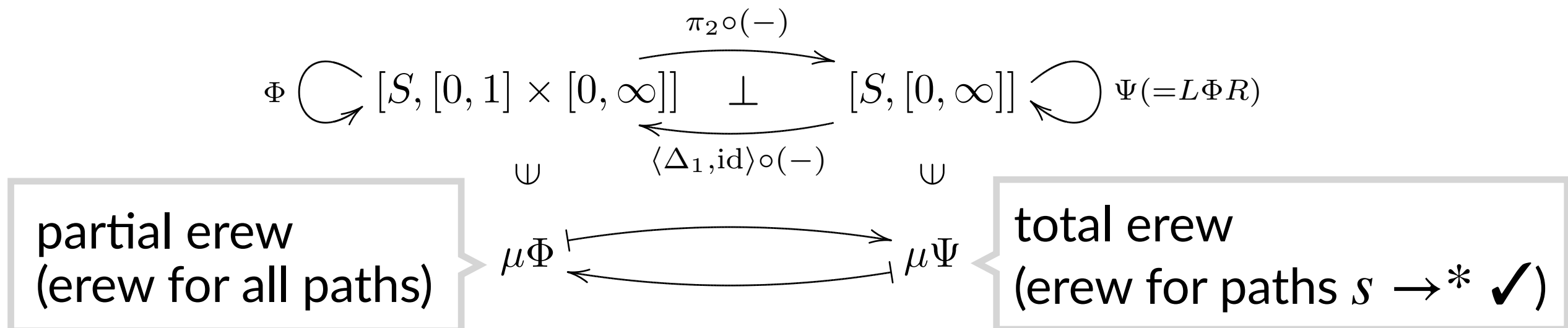
Prop. Assume R is full and faithful.

If $\iota_\Phi \in \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}}$ then initial alg. corresp. for Φ and $L\Phi R$ holds.

We call $\iota_\Phi \in \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}}$ ($\Leftrightarrow \mu\Phi \in \mathbb{C}_{\text{fix}}$) the **global reachability condition of ι_Φ** .

E.g. total/partial expected rewards of MCs.

$\iota_\Phi \in \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}}$ ($\Leftrightarrow \mu\Phi \in \mathbb{C}_{\text{fix}}$) is the almost surely reachability.



Outline

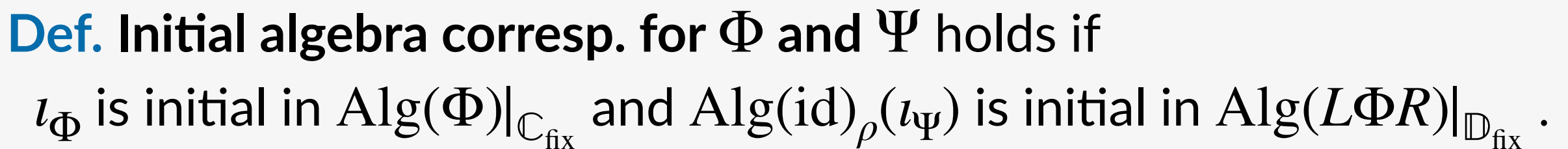
1. Equivalence of subcategories of algebras



2. Initial algebra corresp. for Φ and $L\Phi R$

3. Initial algebra corresp. for Φ and Ψ with a step

Initial algebra correspondence by chains



Initial algebra corresp. for Φ and Ψ with a step

Global reachability condition

Prop. Assume $\mathbb{D}$ has all colimits of chains and L is full and faithful. The initial alg. corresp. for Φ and Ψ holds if

1. $\bar{\rho}$ is an isomorphism.
2. The initial chain of Ψ converges.
3. The initial chain of Ψ lies in $\mathbb{D}_{\text{fix}}$ ($\Leftarrow \mathbb{D}_{\text{fix}}$ is downward closed and $l_{\Psi} \in \text{Alg}(\Psi)|_{\mathbb{D}_{\text{fix}}}$).

We call $l_{\Psi} \in \text{Alg}(\Psi)|_{\mathbb{D}_{\text{fix}}}$ the **global reachability condition** of l_{Ψ} .

E.g. For an NFA, a corresp. btw. num of accepting paths and that of acceptance of words under unambiguity of the NFA.

E.g. A corresp. btw. two probabilistic semantics under unambiguity of NFAs.

Conclusions & other results

- General framework for global reachability conditions ensuring init. alg. correspondence via adjoints.

$$\begin{array}{ccc}
 \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}} & \simeq & \text{Alg}(L\Phi R)|_{\mathbb{D}_{\text{fix}}} \\
 \Psi & & \Psi \\
 \downarrow \iota_{\Phi} & \xrightarrow{\text{Alg}(L)} & \downarrow \iota_{L\Phi R} \\
 & \xleftarrow{\text{Alg}(R)} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Alg}(\Phi)|_{\mathbb{C}_{\text{fix}}} & \simeq & \text{Alg}(L\Phi R)|_{\mathbb{D}_{\text{fix}}} \\
 \Psi & & \Psi \\
 \downarrow \iota_{\Phi} & \xrightarrow{\text{Alg}(L)} & \downarrow \text{Alg}(\text{id})_{\rho}(\iota_{\Psi}) \\
 & \xleftarrow{\text{Alg}(R)} &
 \end{array}$$

- e.g. almost surely reachability, unambiguity of automata
- Fibrational init alg. correspondence
- Construction of predicate transformers for our framework
 - We can naturally derive predicate transformers for total & partial erew of MDP from that of MC